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# Connectedness Conditions Using Polygonal Arcs

Yatsuka Nakamura
Shinshu University
Nagano

Jarosław Kotowicz
Warsaw University
Białystok

**Summary.** A concept of special polygonal arc joining two different points is defined. Any two points in a ball can be connected by this kind of arc, and that is also true for any region in $\mathcal{E}^2_{\mathrm{T}}$.

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The articles [12], [14], [2], [8], [1], [15], [4], [3], [13], [11], [10], [9], [5], [6], and [7] provide the notation and terminology for this paper.

For simplicity, we use the following convention: $P$, $P_1$, $P_2$, $R$ denote subsets of $\mathcal{E}^2_{\mathrm{T}}$, $W$ denotes a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, $p$, $p_1$, $p_2$, $q$ denote points of $\mathcal{E}^2_{\mathrm{T}}$, $f$, $h$ denote finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, $r$ denotes a real number, $u$ denotes a point of $\mathcal{E}^2$, and $n$, $i$ denote natural numbers.

The following proposition is true

(1) Let $D$ be a non empty set, $f$ be a finite sequence of elements of $D$, and $p$ be an element of $D$. Then $(f \frown \langle p \rangle)_{\mathrm{len}\, f+1} = p$.

Let us consider $P$, $p$, $q$. We say that $P$ is a special polygonal arc joining $p$ and $q$ if and only if:

(Def. 1) There exists $f$ such that $f$ is a special sequence and $P = \widetilde{\mathcal{L}}(f)$ and $p = f_1$ and $q = f_{\mathrm{len}\, f}$.

Let us consider $P$. We say that $P$ is special polygon if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exist $p_1$, $p_2$, $P_1$, $P_2$ such that

$p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$ and $P_1$ is a special polygonal arc joining $p_1$ and $p_2$ and $P_2$ is a special polygonal arc joining $p_1$ and $p_2$ and $P_1 \cap P_2 = \{p_1, p_2\}$ and $P = P_1 \cup P_2$.

We introduce $P$ is a special polygon as a synonym of $P$ is special polygon.

Let $P$ be a subset of $\mathcal{E}^2_{\mathrm{T}}$. We say that $P$ is region if and only if:

(Def. 3) $P$ is open and connected.

We introduce $P$ is a region as a synonym of $P$ is region.

Next we state a number of propositions:

(2) If $P$ is a special polygonal arc joining $p$ and $q$, then $P$ is a special polygonal arc.

(3) If $W$ is a special polygonal arc joining $p$ and $q$, then $W$ is an arc from $p$ to $q$.

(4) If $W$ is a special polygonal arc joining $p$ and $q$, then $p \in W$ and $q \in W$.

(5) If $P$ is a special polygonal arc joining $p$ and $q$, then $p \neq q$.

(6) If $W$ is a special polygon, then $W$ is a simple closed curve.

(7) Suppose $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $f = \langle p, [p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}], q\rangle$. Then $f$ is a special sequence and $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining $p$ and $q$ and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(8) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $f = \langle p, [\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}], q\rangle$. Then $f$ is a special sequence and $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining $p$ and $q$ and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(9) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $[p_{\mathbf{1}}, q_{\mathbf{2}}] \in \mathrm{Ball}(u,r)$ and $f = \langle p, [p_{\mathbf{1}}, q_{\mathbf{2}}], q\rangle$. Then $f$ is a special sequence and $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining $p$ and $q$ and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(10) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $[q_{\mathbf{1}}, p_{\mathbf{2}}] \in \mathrm{Ball}(u,r)$ and $f = \langle p, [q_{\mathbf{1}}, p_{\mathbf{2}}], q\rangle$. Then $f$ is a special sequence and $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining $p$ and $q$ and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(11) If $p \neq q$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$, then there exists $P$ such that $P$ is a special polygonal arc joining $p$ and $q$ and $P \subseteq \mathrm{Ball}(u,r)$.

(12) Suppose $p \neq f_1$ and $(f_1)_{\mathbf{2}} = p_{\mathbf{2}}$ and $f$ is a special sequence and $p \in \mathcal{L}(f,1)$ and $h = \langle f_1, [\frac{(f_1)_{\mathbf{1}}+p_{\mathbf{1}}}{2}, (f_1)_{\mathbf{2}}], p\rangle$. Then

(i) $h$ is a special sequence,

(ii) $h_1 = f_1$,

(iii) $h_{\mathrm{len}\, h} = p$,

(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $p$,

(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$, and

(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}1) \cup \mathcal{L}(f_1, p)$.

(13) Suppose $p \neq f_1$ and $(f_1)_{\mathbf{1}} = p_{\mathbf{1}}$ and $f$ is a special sequence and $p \in \mathcal{L}(f,1)$ and $h = \langle f_1, [(f_1)_{\mathbf{1}}, \frac{(f_1)_{\mathbf{2}}+p_{\mathbf{2}}}{2}], p\rangle$. Then

(i) $h$ is a special sequence,

(ii) $h_1 = f_1$,

(iii) $h_{\mathrm{len}\, h} = p$,

(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $p$,

(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$, and

(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}1) \cup \mathcal{L}(f_1, p)$.

(14) Suppose $p \neq f_1$ and $f$ is a special sequence and $i \in \mathrm{dom}\, f$ and $i+1 \in \mathrm{dom}\, f$ and $i > 1$ and $p \in \mathcal{L}(f,i)$ and $p \neq f_i$ and $p \neq f_{i+1}$ and $h = (f{\restriction}i) ^\frown \langle p\rangle$. Then

(i) $h$ is a special sequence,

(ii) $h_1 = f_1$,

(iii) $h_{\mathrm{len}\, h} = p$,

(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $p$,

(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$, and

(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}i) \cup \mathcal{L}(f_i, p)$.

(15) Suppose $f_2 \neq f_1$ and $f$ is a special sequence and $(f_2)_{\mathbf{2}} = (f_1)_{\mathbf{2}}$ and $h = \langle f_1, [\frac{(f_1)_{\mathbf{1}}+(f_2)_{\mathbf{1}}}{2}, (f_1)_{\mathbf{2}}], f_2\rangle$. Then $h$ is a special sequence and $h_1 = f_1$ and $h_{\mathrm{len}\, h} = f_2$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $f_2$ and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}1) \cup \mathcal{L}(f_1, f_2)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}2) \cup \mathcal{L}(f_2, f_2)$.

(16) Suppose $f_2 \neq f_1$ and $f$ is a special sequence and $(f_2)_{\mathbf{1}} = (f_1)_{\mathbf{1}}$ and $h = \langle f_1, [(f_1)_{\mathbf{1}}, \frac{(f_1)_{\mathbf{2}}+(f_2)_{\mathbf{2}}}{2}], f_2\rangle$. Then $h$ is a special sequence and $h_1 = f_1$ and $h_{\operatorname{len} h} = f_2$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $f_2$ and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}1) \cup \mathcal{L}(f_1, f_2)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}2) \cup \mathcal{L}(f_2, f_2)$.

(17) Suppose $f_i \neq f_1$ and $f$ is a special sequence and $i > 2$ and $i \in \operatorname{dom} f$ and $h = f{\restriction}i$. Then

(i) $h$ is a special sequence,

(ii) $h_1 = f_1$,

(iii) $h_{\operatorname{len} h} = f_i$,

(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $f_i$,

(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$, and

(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}i) \cup \mathcal{L}(f_i, f_i)$.

(18) Suppose $p \neq f_1$ and $f$ is a special sequence and $p \in \mathcal{L}(f, n)$. Then there exists $h$ such that

(i) $h$ is a special sequence,

(ii) $h_1 = f_1$,

(iii) $h_{\operatorname{len} h} = p$,

(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $p$,

(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$, and

(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}n) \cup \mathcal{L}(f_n, p)$.

(19) Suppose $p \neq f_1$ and $f$ is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$. Then there exists $h$ such that $h$ is a special sequence and $h_1 = f_1$ and $h_{\operatorname{len} h} = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $p$ and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$.

(20) Suppose that $p_{\mathbf{1}} = (f_{\operatorname{len} f})_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq (f_{\operatorname{len} f})_{\mathbf{2}}$ or $p_{\mathbf{1}} \neq (f_{\operatorname{len} f})_{\mathbf{1}}$ and $p_{\mathbf{2}} = (f_{\operatorname{len} f})_{\mathbf{2}}$ and $f_1 \notin \operatorname{Ball}(u,r)$ and $f_{\operatorname{len} f} \in \operatorname{Ball}(u,r)$ and $p \in \operatorname{Ball}(u,r)$ and $f$ is a special sequence and $\mathcal{L}(f_{\operatorname{len} f}, p) \cap \widetilde{\mathcal{L}}(f) = \{f_{\operatorname{len} f}\}$ and $h = f \frown \langle p\rangle$. Then $h$ is a special sequence and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $p$ and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \operatorname{Ball}(u,r)$.

(21) Suppose that $f_1 \notin \operatorname{Ball}(u,r)$ and $f_{\operatorname{len} f} \in \operatorname{Ball}(u,r)$ and $p \in \operatorname{Ball}(u,r)$ and $[p_{\mathbf{1}}, (f_{\operatorname{len} f})_{\mathbf{2}}] \in \operatorname{Ball}(u,r)$ and $f$ is a special sequence and $p_{\mathbf{1}} \neq (f_{\operatorname{len} f})_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq (f_{\operatorname{len} f})_{\mathbf{2}}$ and $h = f \frown \langle [p_{\mathbf{1}}, (f_{\operatorname{len} f})_{\mathbf{2}}], p\rangle$ and $(\mathcal{L}(f_{\operatorname{len} f}, [p_{\mathbf{1}}, (f_{\operatorname{len} f})_{\mathbf{2}}]) \cup \mathcal{L}([p_{\mathbf{1}}, (f_{\operatorname{len} f})_{\mathbf{2}}], p)) \cap \widetilde{\mathcal{L}}(f) = \{f_{\operatorname{len} f}\}$. Then $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $p$ and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \operatorname{Ball}(u,r)$.

(22) Suppose that $f_1 \notin \operatorname{Ball}(u,r)$ and $f_{\operatorname{len} f} \in \operatorname{Ball}(u,r)$ and $p \in \operatorname{Ball}(u,r)$ and $[(f_{\operatorname{len} f})_{\mathbf{1}}, p_{\mathbf{2}}] \in \operatorname{Ball}(u,r)$ and $f$ is a special sequence and $p_{\mathbf{1}} \neq (f_{\operatorname{len} f})_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq (f_{\operatorname{len} f})_{\mathbf{2}}$ and $h = f \frown \langle [(f_{\operatorname{len} f})_{\mathbf{1}}, p_{\mathbf{2}}], p\rangle$ and $(\mathcal{L}(f_{\operatorname{len} f}, [(f_{\operatorname{len} f})_{\mathbf{1}}, p_{\mathbf{2}}]) \cup \mathcal{L}([(f_{\operatorname{len} f})_{\mathbf{1}}, p_{\mathbf{2}}], p)) \cap \widetilde{\mathcal{L}}(f) = \{f_{\operatorname{len} f}\}$. Then $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $p$ and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \operatorname{Ball}(u,r)$.

(23) Suppose $f_1 \notin \operatorname{Ball}(u,r)$ and $f_{\operatorname{len} f} \in \operatorname{Ball}(u,r)$ and $p \in \operatorname{Ball}(u,r)$ and $f$ is a special sequence and $p \notin \widetilde{\mathcal{L}}(f)$. Then there exists $h$ such that $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining $f_1$ and $p$ and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \operatorname{Ball}(u,r)$.

(24) Let given $R$, $p$, $p_1$, $p_2$, $P$, $r$, $u$. Suppose $p \neq p_1$ and $P$ is a special polygonal arc joining $p_1$ and $p_2$ and $P \subseteq R$ and $p \in \operatorname{Ball}(u,r)$ and $p_2 \in \operatorname{Ball}(u,r)$ and $\operatorname{Ball}(u,r) \subseteq R$. Then there exists a subset $P_1$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $P_1$ is a special polygonal arc joining $p_1$ and $p$ and $P_1 \subseteq R$.

In the sequel $P$, $R$ are subsets of $\mathcal{E}^2_{\mathrm{T}}$.

We now state several propositions:

(25) Let given $p$. Suppose that

(i) $R$ is a region, and

(ii) $P = \{q : q \neq p \wedge q \in R \wedge \neg \bigvee_{P_1 : \text{subset of } \mathcal{E}^2_{\mathrm{T}}} (P_1$ is a special polygonal arc joining $p$ and $q \wedge P_1 \subseteq R)\}$.

Then $P$ is open.

(26) Suppose that

(i) $R$ is a region,

(ii) $p \in R$, and

(iii) $P = \{q : q = p \vee \bigvee_{P_1 : \text{subset of } \mathcal{E}^2_{\mathrm{T}}} (P_1$ is a special polygonal arc joining $p$ and $q \wedge P_1 \subseteq R)\}$.

Then $P$ is open.

(27) Suppose $p \in R$ and $P = \{q : q = p \vee \bigvee_{P_1 : \text{subset of } \mathcal{E}^2_{\mathrm{T}}} (P_1$ is a special polygonal arc joining $p$ and $q \wedge P_1 \subseteq R)\}$. Then $P \subseteq R$.

(28) Suppose that

(i) $R$ is a region,

(ii) $p \in R$, and

(iii) $P = \{q : q = p \vee \bigvee_{P_1 : \text{subset of } \mathcal{E}^2_{\mathrm{T}}} (P_1$ is a special polygonal arc joining $p$ and $q \wedge P_1 \subseteq R)\}$.

Then $R \subseteq P$.

(29) Suppose that

(i) $R$ is a region,

(ii) $p \in R$, and

(iii) $P = \{q : q = p \vee \bigvee_{P_1 : \text{subset of } \mathcal{E}^2_{\mathrm{T}}} (P_1$ is a special polygonal arc joining $p$ and $q \wedge P_1 \subseteq R)\}$.

Then $R = P$.

(30) If $R$ is a region and $p \in R$ and $q \in R$ and $p \neq q$, then there exists $P$ such that $P$ is a special polygonal arc joining $p$ and $q$ and $P \subseteq R$.

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