JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Relations of Tolerance[1]

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Summary. Introduces notions of relations of tolerance, tolerance set and neighbourhood of an element. The basic properties of relations of tolerance are proved.

MML Identifier: TOLER_1.

WWW: http://mizar.org/JFM/Vol2/toler_1.html

The articles [5], [3], [6], [7], [8], [9], [2], [1], and [4] provide the notation and terminology for this paper.

In this paper X, Y, Z, x, y are sets.

The following propositions are true:

(1) $\mathrm{field}\,\emptyset = \emptyset$.

(2) $\emptyset$ is reflexive.

(3) $\emptyset$ is symmetric.

(4) $\emptyset$ is irreflexive.

(5) $\emptyset$ is antisymmetric.

(6) $\emptyset$ is asymmetric.

(7) $\emptyset$ is connected.

(8) $\emptyset$ is strongly connected.

(9) $\emptyset$ is transitive.

Let us observe that $\emptyset$ is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, connected, strongly connected, and transitive.

Let us consider X. We introduce ∇_X as a synonym of ∇_X.

Let R be a binary relation and let Y be a set. Then $R\,|^2\,Y$ is a relation between Y and Y.

Next we state several propositions:

(12)[1] $\mathrm{dom}(\nabla_X) = X$.

(13) $\mathrm{rng}(\nabla_X) = X$.

(15)[2] For all x, y such that $x \in X$ and $y \in X$ holds $\langle x, y \rangle \in \nabla_X$.

[1]Supported by Philippe le Hodey Foundation. This work had been done on Mizar Workshop '89 (Foudrain, France) in Summer '89.

[1] The propositions (10) and (11) have been removed.

[2] The proposition (14) has been removed.

(16) For all x, y such that $x \in \mathrm{field}(\nabla_X)$ and $y \in \mathrm{field}(\nabla_X)$ holds $\langle x, y \rangle \in \nabla_X$.

(19)[3] ∇_X is strongly connected.

(21)[4] ∇_X is connected.

In the sequel T, R are tolerances of X.
We now state several propositions:

(24)[5] For every tolerance T of X holds $\mathrm{dom}\, T = X$.

(25) For every tolerance T of X holds $\mathrm{rng}\, T = X$.

(27)[6] For every total reflexive binary relation T on X holds $x \in X$ iff $\langle x, x \rangle \in T$.

(28) Every tolerance of X is reflexive in X.

(29) Every tolerance of X is symmetric in X.

(32)[7] For every relation R between X and Y such that R is symmetric holds $R\,|^2\, Z$ is symmetric.

Let us consider X, T and let Y be a subset of X. Then $T\,|^2\, Y$ is a tolerance of Y.
The following proposition is true

(33) If $Y \subseteq X$, then $T\,|^2\, Y$ is a tolerance of Y.

Let us consider X and let T be a tolerance of X. A set is called a set of mutually elements w.r.t. T if:

(Def. 3)[8] For all x, y such that $x \in$ it and $y \in$ it holds $\langle x, y \rangle \in T$.

We now state the proposition

(34) $\emptyset$ is a set of mutually elements w.r.t. T.

Let us consider X, let T be a tolerance of X, and let I_1 be a set of mutually elements w.r.t. T. We say that I_1 is tolerance class-like if and only if:

(Def. 4) For every x such that $x \notin I_1$ and $x \in X$ there exists y such that $y \in I_1$ and $\langle x, y \rangle \notin T$.

Let us consider X and let T be a tolerance of X. Note that there exists a set of mutually elements w.r.t. T which is tolerance class-like.

Let us consider X and let T be a tolerance of X. A tolerance class of T is a tolerance class-like set of mutually elements w.r.t. T.

One can prove the following propositions:

(38)[9] For every tolerance T of X such that $\emptyset$ is a tolerance class of T holds $T = \emptyset$.

(39) $\emptyset$ is a tolerance of $\emptyset$.

(40) For all x, y such that $\langle x, y \rangle \in T$ holds $\{x, y\}$ is a set of mutually elements w.r.t. T.

(41) For every x such that $x \in X$ holds $\{x\}$ is a set of mutually elements w.r.t. T.

(42) Let given Y, Z. Suppose Y is a set of mutually elements w.r.t. T and Z is a set of mutually elements w.r.t. T. Then $Y \cap Z$ is a set of mutually elements w.r.t. T.

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[3] The propositions (17) and (18) have been removed.
[4] The proposition (20) has been removed.
[5] The propositions (22) and (23) have been removed.
[6] The proposition (26) has been removed.
[7] The propositions (30) and (31) have been removed.
[8] The definitions (Def. 1) and (Def. 2) have been removed.
[9] The propositions (35)–(37) have been removed.

(43) If Y is a set of mutually elements w.r.t. T, then $Y \subseteq X$.

(45)[10] For every set Y of mutually elements w.r.t. T there exists a tolerance class Z of T such that $Y \subseteq Z$.

(46) For all x, y such that $\langle x, y\rangle \in T$ there exists a tolerance class Z of T such that $x \in Z$ and $y \in Z$.

(47) For every x such that $x \in X$ there exists a tolerance class Z of T such that $x \in Z$.

(49)[11] $T \subseteq \nabla_X$.

(50) $\mathrm{id}_X \subseteq T$.

The scheme *ToleranceEx* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a tolerance T of $\mathcal{A}$ such that for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{A}$ holds $\langle x, y\rangle \in T$ iff $\mathcal{P}[x,y]$

provided the parameters satisfy the following conditions:

- For every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x,x]$, and
- For all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{A}$ and $\mathcal{P}[x,y]$ holds $\mathcal{P}[y,x]$.

Next we state three propositions:

(51) Let given Y. Then there exists a tolerance T of $\bigcup Y$ such that for every Z if $Z \in Y$, then Z is a set of mutually elements w.r.t. T.

(52) Let Y be a set and T, R be tolerances of $\bigcup Y$. Suppose that

(i) for all x, y holds $\langle x, y\rangle \in T$ iff there exists Z such that $Z \in Y$ and $x \in Z$ and $y \in Z$, and

(ii) for all x, y holds $\langle x, y\rangle \in R$ iff there exists Z such that $Z \in Y$ and $x \in Z$ and $y \in Z$.

Then $T = R$.

(53) Let T, R be tolerances of X. Suppose that for every Z holds Z is a tolerance class of T iff Z is a tolerance class of R. Then $T = R$.

Let us consider X, let T be a tolerance of X, and let us consider x. We introduce $\mathrm{neighbourhood}(x,T)$ as a synonym of $[x]_T$.

The following three propositions are true:

(54) For every set y holds $y \in \mathrm{neighbourhood}(x,T)$ iff $\langle x, y\rangle \in T$.

(58)[12] For every Y such that for every set Z holds $Z \in Y$ iff $x \in Z$ and Z is a tolerance class of T holds $\mathrm{neighbourhood}(x,T) = \bigcup Y$.

(59) Let given Y. Suppose that for every Z holds $Z \in Y$ iff $x \in Z$ and Z is a set of mutually elements w.r.t. T. Then $\mathrm{neighbourhood}(x,T) = \bigcup Y$.

Let us consider X and let T be a tolerance of X. The functor $\mathrm{TolSets}\,T$ yielding a set is defined by:

(Def. 6)[13] For every Y holds $Y \in \mathrm{TolSets}\,T$ iff Y is a set of mutually elements w.r.t. T.

The functor $\mathrm{TolClasses}\,T$ yielding a set is defined as follows:

(Def. 7) For every Y holds $Y \in \mathrm{TolClasses}\,T$ iff Y is a tolerance class of T.

The following propositions are true:

[10] The proposition (44) has been removed.
[11] The proposition (48) has been removed.
[12] The propositions (55)–(57) have been removed.
[13] The definition (Def. 5) has been removed.

(64)[14] If $\mathrm{TolClasses}\,R \subseteq \mathrm{TolClasses}\,T$, then $R \subseteq T$.

(65) For all tolerances T, R of X such that $\mathrm{TolClasses}\,T = \mathrm{TolClasses}\,R$ holds $T = R$.

(66) $\bigcup \mathrm{TolClasses}\,T = X$.

(67) $\bigcup \mathrm{TolSets}\,T = X$.

(68) If for every x such that $x \in X$ holds $\mathrm{neighbourhood}(x, T)$ is a set of mutually elements w.r.t. T, then T is transitive.

(69) If T is transitive, then for every x such that $x \in X$ holds $\mathrm{neighbourhood}(x, T)$ is a tolerance class of T.

(70) For every x and for every tolerance class Y of T such that $x \in Y$ holds $Y \subseteq \mathrm{neighbourhood}(x, T)$.

(71) $\mathrm{TolSets}\,R \subseteq \mathrm{TolSets}\,T$ iff $R \subseteq T$.

(72) $\mathrm{TolClasses}\,T \subseteq \mathrm{TolSets}\,T$.

(73) If for every x such that $x \in X$ holds $\mathrm{neighbourhood}(x, R) \subseteq \mathrm{neighbourhood}(x, T)$, then $R \subseteq T$.

(74) $T \subseteq T \cdot T$.

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Received September 20, 1990

Published January 2, 2004

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[14] The propositions (60)–(63) have been removed.