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Inst. of Computer Science, Univ. of Białystok

Continuity of Mappings over the Union of Subspaces

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X and Y be topological spaces and let X_1 and X_2 be subspaces of X. Let $f : X_1 \cup X_2 \to Y$ be a mapping defined on the union of X_1 and X_2 such that the restriction mappings $f_{|X_1}$ and $f_{|X_2}$ are continuous. It is well known that if X_1 and X_2 are both open (closed) subspaces of X, then f is continuous (see e.g. [7, p.106]).

The aim is to show, using Mizar System, the following theorem (see Section 5): *If X_1 and X_2 are weakly separated, then f is continuous* (compare also [14, p.358] for related results). This theorem generalizes the preceding one because if X_1 and X_2 are both open (closed), then these subspaces are weakly separated (see [6]). However, the following problem remains open.

Problem 1. Characterize the class of pairs of subspaces X_1 and X_2 of a topological space X such that $(*)$ for any topological space Y and for any mapping $f : X_1 \cup X_2 \to Y$, f is continuous if the restrictions $f_{|X_1}$ and $f_{|X_2}$ are continuous.

In some special case we have the following characterization: *X_1 and X_2 are separated iff X_1 misses X_2 and the condition $(*)$ is fulfilled.* In connection with this fact we hope that the following specification of the preceding problem has an affirmative answer.

Problem 2. Suppose the condition $(*)$ is fulfilled. Must X_1 and X_2 be weakly separated ?

Note that in the last section the concept of the union of two mappings is introduced and studied. In particular, all results presented above are reformulated using this notion. In the remaining sections we introduce concepts needed for the formulation and the proof of theorems on properties of continuous mappings, restriction mappings and modifications of the topology.

MML Identifier: `TMAP_1`.

WWW: `http://mizar.org/JFM/Vol4/tmap_1.html`

The articles [11], [4], [13], [15], [1], [3], [2], [5], [10], [9], [12], [8], and [6] provide the notation and terminology for this paper.

1. Set-Theoretic Preliminaries

Let X be a non empty topological space and let X_1, X_2 be non empty subspaces of X. Observe that $X_1 \cup X_2$ is topological space-like.

In the sequel A, B denote non empty sets.

The following propositions are true:

(1) Let f be a function from A into B, A_0 be a subset of A, and B_0 be a subset of B. Then $f^\circ A_0 \subseteq B_0$ if and only if $A_0 \subseteq f^{-1}(B_0)$.

(2) Let f be a function from A into B, A_0 be a non empty subset of A, and f_0 be a function from A_0 into B. If for every element c of A such that $c \in A_0$ holds $f(c) = f_0(c)$, then $f{\restriction}A_0 = f_0$.

(4)[1] Let f be a function from A into B, A_0 be a non empty subset of A, and C be a subset of A. If $C \subseteq A_0$, then $f^\circ C = (f \restriction A_0)^\circ C$.

(5) Let f be a function from A into B, A_0 be a non empty subset of A, and D be a subset of B. If $f^{-1}(D) \subseteq A_0$, then $f^{-1}(D) = (f \restriction A_0)^{-1}(D)$.

Let A, B be non empty sets, let A_1, A_2 be non empty subsets of A, let f_1 be a function from A_1 into B, and let f_2 be a function from A_2 into B. Let us assume that $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$. The functor $f_1 \cup f_2$ yielding a function from $A_1 \cup A_2$ into B is defined by:

(Def. 1) $(f_1 \cup f_2) \restriction A_1 = f_1$ and $(f_1 \cup f_2) \restriction A_2 = f_2$.

One can prove the following proposition

(6) Let A, B be non empty sets and A_1, A_2 be non empty subsets of A. Suppose A_1 misses A_2. Let f_1 be a function from A_1 into B and f_2 be a function from A_2 into B. Then $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ and $(f_1 \cup f_2) \restriction A_1 = f_1$ and $(f_1 \cup f_2) \restriction A_2 = f_2$.

We use the following convention: A, B are non empty sets and A_1, A_2, A_3 are non empty subsets of A.

The following four propositions are true:

(7) Let g be a function from $A_1 \cup A_2$ into B, g_1 be a function from A_1 into B, and g_2 be a function from A_2 into B. If $g \restriction A_1 = g_1$ and $g \restriction A_2 = g_2$, then $g = g_1 \cup g_2$.

(8) For every function f_1 from A_1 into B and for every function f_2 from A_2 into B such that $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ holds $f_1 \cup f_2 = f_2 \cup f_1$.

(9) Let A_{12}, A_{23} be non empty subsets of A. Suppose $A_{12} = A_1 \cup A_2$ and $A_{23} = A_2 \cup A_3$. Let f_1 be a function from A_1 into B, f_2 be a function from A_2 into B, and f_3 be a function from A_3 into B. Suppose $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ and $f_2 \restriction (A_2 \cap A_3) = f_3 \restriction (A_2 \cap A_3)$ and $f_1 \restriction (A_1 \cap A_3) = f_3 \restriction (A_1 \cap A_3)$. Let f_{12} be a function from A_{12} into B and f_{23} be a function from A_{23} into B. If $f_{12} = f_1 \cup f_2$ and $f_{23} = f_2 \cup f_3$, then $f_{12} \cup f_3 = f_1 \cup f_{23}$.

(10) Let f_1 be a function from A_1 into B and f_2 be a function from A_2 into B such that $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$. Then

(i) A_1 is a subset of A_2 iff $f_1 \cup f_2 = f_2$, and

(ii) A_2 is a subset of A_1 iff $f_1 \cup f_2 = f_1$.

2. Selected Properties of Subspaces of Topological Spaces

In the sequel X is a non empty topological space.

Next we state four propositions:

(11) For every non empty subspace X_0 of X holds the topological structure of X_0 is a strict subspace of X.

(12) Let X_1, X_2 be non empty topological spaces. Suppose $X_1 =$ the topological structure of X_2. Then X_1 is a subspace of X if and only if X_2 is a subspace of X.

(13) Let X_1, X_2 be non empty topological spaces. Suppose $X_2 =$ the topological structure of X_1. Then X_1 is a closed subspace of X if and only if X_2 is a closed subspace of X.

(14) Let X_1, X_2 be non empty topological spaces. Suppose $X_2 =$ the topological structure of X_1. Then X_1 is an open subspace of X if and only if X_2 is an open subspace of X.

In the sequel X_1, X_2 are non empty subspaces of X.

The following propositions are true:

[1] The proposition (3) has been removed.

(15) If X_1 is a subspace of X_2, then for every point x_1 of X_1 there exists a point x_2 of X_2 such that $x_2 = x_1$.

(16) For every point x of $X_1 \cup X_2$ holds there exists a point x_1 of X_1 such that $x_1 = x$ or there exists a point x_2 of X_2 such that $x_2 = x$.

(17) Suppose X_1 meets X_2. Let x be a point of $X_1 \cap X_2$. Then there exists a point x_1 of X_1 such that $x_1 = x$ and there exists a point x_2 of X_2 such that $x_2 = x$.

(18) Let x be a point of $X_1 \cup X_2$, F_1 be a subset of X_1, and F_2 be a subset of X_2. Suppose F_1 is closed and $x \in F_1$ and F_2 is closed and $x \in F_2$. Then there exists a subset H of $X_1 \cup X_2$ such that H is closed and $x \in H$ and $H \subseteq F_1 \cup F_2$.

(19) Let x be a point of $X_1 \cup X_2$, U_1 be a subset of X_1, and U_2 be a subset of X_2. Suppose U_1 is open and $x \in U_1$ and U_2 is open and $x \in U_2$. Then there exists a subset V of $X_1 \cup X_2$ such that V is open and $x \in V$ and $V \subseteq U_1 \cup U_2$.

(20) Let x be a point of $X_1 \cup X_2$, x_1 be a point of X_1, and x_2 be a point of X_2. Suppose $x_1 = x$ and $x_2 = x$. Let A_1 be a neighbourhood of x_1 and A_2 be a neighbourhood of x_2. Then there exists a subset V of $X_1 \cup X_2$ such that V is open and $x \in V$ and $V \subseteq A_1 \cup A_2$.

(21) Let x be a point of $X_1 \cup X_2$, x_1 be a point of X_1, and x_2 be a point of X_2. Suppose $x_1 = x$ and $x_2 = x$. Let A_1 be a neighbourhood of x_1 and A_2 be a neighbourhood of x_2. Then there exists a neighbourhood A of x such that $A \subseteq A_1 \cup A_2$.

In the sequel X_0, X_1, X_2, Y_1, Y_2 are non empty subspaces of X.
The following propositions are true:

(22) If X_0 is a subspace of X_1, then X_0 meets X_1 and X_1 meets X_0.

(23) If X_0 is a subspace of X_1 and if X_0 meets X_2 or X_2 meets X_0, then X_1 meets X_2 and X_2 meets X_1.

(24) If X_0 is a subspace of X_1 and if X_1 misses X_2 or X_2 misses X_1, then X_0 misses X_2 and X_2 misses X_0.

(25) $X_0 \cup X_0 =$ the topological structure of X_0.

(26) $X_0 \cap X_0 =$ the topological structure of X_0.

(27) If Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2, then $Y_1 \cup Y_2$ is a subspace of $X_1 \cup X_2$.

(28) If Y_1 meets Y_2 and Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2, then $Y_1 \cap Y_2$ is a subspace of $X_1 \cap X_2$.

(29) If X_1 is a subspace of X_0 and X_2 is a subspace of X_0, then $X_1 \cup X_2$ is a subspace of X_0.

(30) If X_1 meets X_2 and X_1 is a subspace of X_0 and X_2 is a subspace of X_0, then $X_1 \cap X_2$ is a subspace of X_0.

(31)(i) If X_1 misses X_0 or X_0 misses X_1 and if X_2 meets X_0 or X_0 meets X_2, then $(X_1 \cup X_2) \cap X_0 = X_2 \cap X_0$ and $X_0 \cap (X_1 \cup X_2) = X_0 \cap X_2$, and

(ii) if X_1 meets X_0 or X_0 meets X_1 and if X_2 misses X_0 or X_0 misses X_2, then $(X_1 \cup X_2) \cap X_0 = X_1 \cap X_0$ and $X_0 \cap (X_1 \cup X_2) = X_0 \cap X_1$.

(32) Suppose X_1 meets X_2. Then

(i) if X_1 is a subspace of X_0, then $X_1 \cap X_2$ is a subspace of $X_0 \cap X_2$, and

(ii) if X_2 is a subspace of X_0, then $X_1 \cap X_2$ is a subspace of $X_1 \cap X_0$.

(33) Suppose X_1 is a subspace of X_0 but X_0 misses X_2 or X_2 misses X_0. Then $X_0 \cap (X_1 \cup X_2) =$ the topological structure of X_1 and $X_0 \cap (X_2 \cup X_1) =$ the topological structure of X_1.

(34) Suppose X_1 meets X_2. Then

(i) if X_1 is a subspace of X_0, then $X_0 \cap X_2$ meets X_1 and $X_2 \cap X_0$ meets X_1, and

(ii) if X_2 is a subspace of X_0, then $X_1 \cap X_0$ meets X_2 and $X_0 \cap X_1$ meets X_2.

(35) Suppose X_1 is a subspace of Y_1 but X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$. Then Y_1 misses X_2 and Y_2 misses X_1.

(36) Suppose that

(i) X_1 is not a subspace of X_2,

(ii) X_2 is not a subspace of X_1,

(iii) $X_1 \cup X_2$ is a subspace of $Y_1 \cup Y_2$,

(iv) $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1, and

(v) $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2.

Then Y_1 meets $X_1 \cup X_2$ and Y_2 meets $X_1 \cup X_2$.

(37) Suppose that X_1 meets X_2 and X_1 is not a subspace of X_2 and X_2 is not a subspace of X_1 and the topological structure of $X = Y_1 \cup Y_2 \cup X_0$ and $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1 and $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2 and $X_0 \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$. Then Y_1 meets $X_1 \cup X_2$ and Y_2 meets $X_1 \cup X_2$.

(38) Suppose that X_1 meets X_2 and X_1 is not a subspace of X_2 and X_2 is not a subspace of X_1 and $X_1 \cup X_2$ is a subspace of $Y_1 \cup Y_2$ and the topological structure of $X = Y_1 \cup Y_2 \cup X_0$ and $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1 and $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2 and $X_0 \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$. Then $Y_1 \cup Y_2$ meets $X_1 \cup X_2$ and X_0 meets $X_1 \cup X_2$.

(39)(i) $X_1 \cup X_2$ meets X_0 iff X_1 meets X_0 or X_2 meets X_0, and

(ii) X_0 meets $X_1 \cup X_2$ iff X_0 meets X_1 or X_0 meets X_2.

(40)(i) $X_1 \cup X_2$ misses X_0 iff X_1 misses X_0 and X_2 misses X_0, and

(ii) X_0 misses $X_1 \cup X_2$ iff X_0 misses X_1 and X_0 misses X_2.

(41) Suppose X_1 meets X_2. Then

(i) if $X_1 \cap X_2$ meets X_0, then X_1 meets X_0 and X_2 meets X_0, and

(ii) if X_0 meets $X_1 \cap X_2$, then X_0 meets X_1 and X_0 meets X_2.

(42) Suppose X_1 meets X_2. Then

(i) if X_1 misses X_0 or X_2 misses X_0, then $X_1 \cap X_2$ misses X_0, and

(ii) if X_0 misses X_1 or X_0 misses X_2, then X_0 misses $X_1 \cap X_2$.

(43) For every closed non empty subspace X_0 of X such that X_0 meets X_1 holds $X_0 \cap X_1$ is a closed subspace of X_1.

(44) For every open non empty subspace X_0 of X such that X_0 meets X_1 holds $X_0 \cap X_1$ is an open subspace of X_1.

(45) Let X_0 be a closed non empty subspace of X. Suppose X_1 is a subspace of X_0 and X_0 misses X_2. Then X_1 is a closed subspace of $X_1 \cup X_2$ and a closed subspace of $X_2 \cup X_1$.

(46) Let X_0 be an open non empty subspace of X. Suppose X_1 is a subspace of X_0 and X_0 misses X_2. Then X_1 is an open subspace of $X_1 \cup X_2$ and an open subspace of $X_2 \cup X_1$.

3. Continuity of Mappings

Let X, Y be non empty topological spaces, let f be a map from X into Y, and let x be a point of X. We say that f is continuous at x if and only if:

(Def. 2) For every neighbourhood G of $f(x)$ there exists a neighbourhood H of x such that $f^\circ H \subseteq G$.

We introduce f is not continuous at x as an antonym of f is continuous at x.

In the sequel X, Y are non empty topological spaces and f is a map from X into Y.

The following propositions are true:

(47) Let x be a point of X. Then f is continuous at x if and only if for every neighbourhood G of $f(x)$ holds $f^{-1}(G)$ is a neighbourhood of x.

(48) Let x be a point of X. Then f is continuous at x if and only if for every subset G of Y such that G is open and $f(x) \in G$ there exists a subset H of X such that H is open and $x \in H$ and $f^\circ H \subseteq G$.

(49) f is continuous iff for every point x of X holds f is continuous at x.

(50) Let X, Y, Z be non empty topological spaces. Suppose the carrier of Y = the carrier of Z and the topology of $Z \subseteq$ the topology of Y. Let f be a map from X into Y and g be a map from X into Z. Suppose $f = g$. Let x be a point of X. If f is continuous at x, then g is continuous at x.

(51) Let X, Y, Z be non empty topological spaces. Suppose the carrier of X = the carrier of Y and the topology of $Y \subseteq$ the topology of X. Let f be a map from X into Z and g be a map from Y into Z. Suppose $f = g$. Let x be a point of X and y be a point of Y. If $x = y$, then if g is continuous at y, then f is continuous at x.

We adopt the following rules: X, Y, Z denote non empty topological spaces, f denotes a map from X into Y, and g denotes a map from Y into Z.

Next we state several propositions:

(52) Let x be a point of X and y be a point of Y. Suppose $y = f(x)$. If f is continuous at x and g is continuous at y, then $g \cdot f$ is continuous at x.

(53) Let y be a point of Y. Suppose f is continuous and g is continuous at y. Let x be a point of X. If $x \in f^{-1}(\{y\})$, then $g \cdot f$ is continuous at x.

(54) For every point x of X such that f is continuous at x and g is continuous holds $g \cdot f$ is continuous at x.

(55) f is a continuous map from X into Y iff for every point x of X holds f is continuous at x.

(56) Let X, Y, Z be non empty topological spaces. Suppose the carrier of Y = the carrier of Z and the topology of $Z \subseteq$ the topology of Y. Then every continuous map from X into Y is a continuous map from X into Z.

(57) Let X, Y, Z be non empty topological spaces. Suppose the carrier of X = the carrier of Y and the topology of $Y \subseteq$ the topology of X. Then every continuous map from Y into Z is a continuous map from X into Z.

Let X, Y be non empty topological spaces, let X_0 be a subspace of X, and let f be a map from X into Y. The functor $f{\restriction}X_0$ yields a map from X_0 into Y and is defined as follows:

(Def. 3) $f{\restriction}X_0 = f{\restriction}$the carrier of X_0.

In the sequel X, Y denote non empty topological spaces, X_0 denotes a non empty subspace of X, and f denotes a map from X into Y.

Next we state several propositions:

(58) For every point x of X such that $x \in$ the carrier of X_0 holds $f(x) = (f \restriction X_0)(x)$.

(59) Let f_0 be a map from X_0 into Y. Suppose that for every point x of X such that $x \in$ the carrier of X_0 holds $f(x) = f_0(x)$. Then $f \restriction X_0 = f_0$.

(60) If the topological structure of $X_0 =$ the topological structure of X, then $f = f \restriction X_0$.

(61) For every subset A of X such that $A \subseteq$ the carrier of X_0 holds $f^\circ A = (f \restriction X_0)^\circ A$.

(62) For every subset B of Y such that $f^{-1}(B) \subseteq$ the carrier of X_0 holds $f^{-1}(B) = (f \restriction X_0)^{-1}(B)$.

(63) For every map g from X_0 into Y there exists a map h from X into Y such that $h \restriction X_0 = g$.

In the sequel f denotes a map from X into Y and X_0 denotes a non empty subspace of X.
Next we state several propositions:

(64) Let x be a point of X and x_0 be a point of X_0. If $x = x_0$, then if f is continuous at x, then $f \restriction X_0$ is continuous at x_0.

(65) Let A be a subset of X, x be a point of X, and x_0 be a point of X_0. Suppose $A \subseteq$ the carrier of X_0 and A is a neighbourhood of x and $x = x_0$. Then f is continuous at x if and only if $f \restriction X_0$ is continuous at x_0.

(66) Let A be a subset of X, x be a point of X, and x_0 be a point of X_0. Suppose A is open and $x \in A$ and $A \subseteq$ the carrier of X_0 and $x = x_0$. Then f is continuous at x if and only if $f \restriction X_0$ is continuous at x_0.

(67) Let X_0 be an open non empty subspace of X, x be a point of X, and x_0 be a point of X_0. If $x = x_0$, then f is continuous at x iff $f \restriction X_0$ is continuous at x_0.

(68) Let f be a continuous map from X into Y and X_0 be a non empty subspace of X. Then $f \restriction X_0$ is a continuous map from X_0 into Y.

(69) Let X, Y, Z be non empty topological spaces, X_0 be a non empty subspace of X, f be a map from X into Y, and g be a map from Y into Z. Then $(g \cdot f) \restriction X_0 = g \cdot (f \restriction X_0)$.

(70) Let X, Y, Z be non empty topological spaces, X_0 be a non empty subspace of X, g be a map from Y into Z, and f be a map from X into Y. If g is continuous and $f \restriction X_0$ is continuous, then $(g \cdot f) \restriction X_0$ is continuous.

(71) Let X, Y, Z be non empty topological spaces, X_0 be a non empty subspace of X, g be a continuous map from Y into Z, and f be a map from X into Y. Suppose $f \restriction X_0$ is a continuous map from X_0 into Y. Then $(g \cdot f) \restriction X_0$ is a continuous map from X_0 into Z.

Let X, Y be non empty topological spaces, let X_0, X_1 be subspaces of X, and let g be a map from X_0 into Y. Let us assume that X_1 is a subspace of X_0. The functor $g \restriction X_1$ yielding a map from X_1 into Y is defined by:

(Def. 4) $g \restriction X_1 = g \restriction$the carrier of X_1.

For simplicity, we use the following convention: X, Y are non empty topological spaces, X_0, X_1 are non empty subspaces of X, f is a map from X into Y, and g is a map from X_0 into Y.
We now state several propositions:

(72) If X_1 is a subspace of X_0, then for every point x_0 of X_0 such that $x_0 \in$ the carrier of X_1 holds $g(x_0) = (g \restriction X_1)(x_0)$.

(73) Suppose X_1 is a subspace of X_0. Let g_1 be a map from X_1 into Y. Suppose that for every point x_0 of X_0 such that $x_0 \in$ the carrier of X_1 holds $g(x_0) = g_1(x_0)$. Then $g \restriction X_1 = g_1$.

(74) $g = g \restriction X_0$.

(75) If X_1 is a subspace of X_0, then for every subset A of X_0 such that $A \subseteq$ the carrier of X_1 holds $g^\circ A = (g{\restriction}X_1)^\circ A$.

(76) If X_1 is a subspace of X_0, then for every subset B of Y such that $g^{-1}(B) \subseteq$ the carrier of X_1 holds $g^{-1}(B) = (g{\restriction}X_1)^{-1}(B)$.

(77) For every map g from X_0 into Y such that $g = f{\restriction}X_0$ holds if X_1 is a subspace of X_0, then $g{\restriction}X_1 = f{\restriction}X_1$.

(78) If X_1 is a subspace of X_0, then $f{\restriction}X_0{\restriction}X_1 = f{\restriction}X_1$.

(79) Let X_0, X_1, X_2 be non empty subspaces of X. Suppose X_1 is a subspace of X_0 and X_2 is a subspace of X_1. Let g be a map from X_0 into Y. Then $g{\restriction}X_1{\restriction}X_2 = g{\restriction}X_2$.

(80) Let f be a map from X into Y, f_0 be a map from X_1 into Y, and g be a map from X_0 into Y. If $X_0 = X$ and $f = g$, then $g{\restriction}X_1 = f_0$ iff $f{\restriction}X_1 = f_0$.

We use the following convention: X_0, X_1, X_2 denote non empty subspaces of X, f denotes a map from X into Y, and g denotes a map from X_0 into Y.

Next we state a number of propositions:

(81) Let x_0 be a point of X_0 and x_1 be a point of X_1. Suppose $x_0 = x_1$. Suppose X_1 is a subspace of X_0 and g is continuous at x_0. Then $g{\restriction}X_1$ is continuous at x_1.

(82) Suppose X_1 is a subspace of X_0. Let x_0 be a point of X_0 and x_1 be a point of X_1. If $x_0 = x_1$, then if $f{\restriction}X_0$ is continuous at x_0, then $f{\restriction}X_1$ is continuous at x_1.

(83) Suppose X_1 is a subspace of X_0. Let A be a subset of X_0, x_0 be a point of X_0, and x_1 be a point of X_1. Suppose $A \subseteq$ the carrier of X_1 and A is a neighbourhood of x_0 and $x_0 = x_1$. Then g is continuous at x_0 if and only if $g{\restriction}X_1$ is continuous at x_1.

(84) Suppose X_1 is a subspace of X_0. Let A be a subset of X_0, x_0 be a point of X_0, and x_1 be a point of X_1. Suppose A is open and $x_0 \in A$ and $A \subseteq$ the carrier of X_1 and $x_0 = x_1$. Then g is continuous at x_0 if and only if $g{\restriction}X_1$ is continuous at x_1.

(85) Suppose X_1 is a subspace of X_0. Let A be a subset of X, x_0 be a point of X_0, and x_1 be a point of X_1. Suppose A is open and $x_0 \in A$ and $A \subseteq$ the carrier of X_1 and $x_0 = x_1$. Then g is continuous at x_0 if and only if $g{\restriction}X_1$ is continuous at x_1.

(86) Suppose X_1 is an open subspace of X_0. Let x_0 be a point of X_0 and x_1 be a point of X_1. If $x_0 = x_1$, then g is continuous at x_0 iff $g{\restriction}X_1$ is continuous at x_1.

(87) Suppose X_1 is an open subspace of X and a subspace of X_0. Let x_0 be a point of X_0 and x_1 be a point of X_1. If $x_0 = x_1$, then g is continuous at x_0 iff $g{\restriction}X_1$ is continuous at x_1.

(88) Suppose the topological structure of $X_1 = X_0$. Let x_0 be a point of X_0 and x_1 be a point of X_1. If $x_0 = x_1$, then if $g{\restriction}X_1$ is continuous at x_1, then g is continuous at x_0.

(89) Let g be a continuous map from X_0 into Y. Suppose X_1 is a subspace of X_0. Then $g{\restriction}X_1$ is a continuous map from X_1 into Y.

(90) Suppose X_1 is a subspace of X_0 and X_2 is a subspace of X_1. Let g be a map from X_0 into Y. Suppose $g{\restriction}X_1$ is a continuous map from X_1 into Y. Then $g{\restriction}X_2$ is a continuous map from X_2 into Y.

(91) For every non empty 1-sorted structure X and for every element x of X holds $\mathrm{id}_X(x) = x$.

(92) Let X be a non empty 1-sorted structure and f be a map from X into X. If for every element x of X holds $f(x) = x$, then $f = \mathrm{id}_X$.

(93) Let X, Y be non empty 1-sorted structures and f be a function from the carrier of X into the carrier of Y. Then $f \cdot \mathrm{id}_X = f$ and $\mathrm{id}_Y \cdot f = f$.

(94) id_X is a continuous map from X into X.

Let X be a non empty topological space and let X_0 be a subspace of X. The functor $\overset{X_0}{\hookrightarrow}$ yields a map from X_0 into X and is defined as follows:

(Def. 6)[2] $\overset{X_0}{\hookrightarrow} = \mathrm{id}_X \restriction X_0$.

We introduce $X_0 \hookrightarrow X$ as a synonym of $\overset{X_0}{\hookrightarrow}$.

One can prove the following four propositions:

(95) For every non empty subspace X_0 of X and for every point x of X such that $x \in$ the carrier of X_0 holds $(\overset{X_0}{\hookrightarrow})(x) = x$.

(96) Let X_0 be a non empty subspace of X and f_0 be a map from X_0 into X. Suppose that for every point x of X such that $x \in$ the carrier of X_0 holds $x = f_0(x)$. Then $\overset{X_0}{\hookrightarrow} = f_0$.

(97) For every non empty subspace X_0 of X and for every map f from X into Y holds $f \restriction X_0 = f \cdot (\overset{X_0}{\hookrightarrow})$.

(98) For every non empty subspace X_0 of X holds $\overset{X_0}{\hookrightarrow}$ is a continuous map from X_0 into X.

4. A Modification of the Topology of Topological Spaces

In the sequel X is a non empty topological space and H, G are subsets of X.

Let us consider X and let A be a subset of X. The A-extension of the topology of X yielding a family of subsets of X is defined by:

(Def. 7) The A-extension of the topology of $X = \{H \cup G \cap A : H \in$ the topology of $X \wedge G \in$ the topology of $X\}$.

We now state several propositions:

(99) For every subset A of X holds the topology of $X \subseteq$ the A-extension of the topology of X.

(100) Let A be a subset of X. Then $\{G \cap A; G$ ranges over subsets of X: $G \in$ the topology of $X\} \subseteq$ the A-extension of the topology of X.

(101) Let A be a subset of X and C, D be subsets of X. Suppose $C \in$ the topology of X and $D \in \{G \cap A; G$ ranges over subsets of X: $G \in$ the topology of $X\}$. Then $C \cup D \in$ the A-extension of the topology of X and $C \cap D \in$ the A-extension of the topology of X.

(102) For every subset A of X holds $A \in$ the A-extension of the topology of X.

(103) Let A be a subset of X. Then $A \in$ the topology of X if and only if the topology of $X =$ the A-extension of the topology of X.

Let X be a non empty topological space and let A be a subset of X. The X modified w.r.t. A yields a strict topological space and is defined as follows:

(Def. 8) The X modified w.r.t. $A = \langle$the carrier of X, the A-extension of the topology of $X\rangle$.

Let X be a non empty topological space and let A be a subset of X. One can verify that the X modified w.r.t. A is non empty.

In the sequel A is a subset of X.

One can prove the following three propositions:

(104)(i) The carrier of the X modified w.r.t. $A =$ the carrier of X, and

(ii) the topology of the X modified w.r.t. $A =$ the A-extension of the topology of X.

(105) For every subset B of the X modified w.r.t. A such that $B = A$ holds B is open.

[2] The definition (Def. 5) has been removed.

(106) Let A be a subset of X. Then A is open if and only if the topological structure of X = the X modified w.r.t. A.

Let X be a non empty topological space and let A be a subset of X. The functor $\mathrm{modid}_{X,A}$ yielding a map from X into the X modified w.r.t. A is defined by:

(Def. 9) $\mathrm{modid}_{X,A} = \mathrm{id}_{\text{the carrier of } X}$.

The following propositions are true:

(107) For every subset A of X such that A is open holds $\mathrm{modid}_{X,A} = \mathrm{id}_X$.

(108) For every point x of X such that $x \notin A$ holds $\mathrm{modid}_{X,A}$ is continuous at x.

(109) Let X_0 be a non empty subspace of X. Suppose the carrier of X_0 misses A. Let x_0 be a point of X_0. Then $\mathrm{modid}_{X,A} \restriction X_0$ is continuous at x_0.

(110) Let X_0 be a non empty subspace of X. Suppose the carrier of $X_0 = A$. Let x_0 be a point of X_0. Then $\mathrm{modid}_{X,A} \restriction X_0$ is continuous at x_0.

(111) Let X_0 be a non empty subspace of X. Suppose the carrier of X_0 misses A. Then $\mathrm{modid}_{X,A} \restriction X_0$ is a continuous map from X_0 into the X modified w.r.t. A.

(112) Let X_0 be a non empty subspace of X. Suppose the carrier of $X_0 = A$. Then $\mathrm{modid}_{X,A} \restriction X_0$ is a continuous map from X_0 into the X modified w.r.t. A.

(113) Let A be a subset of X. Then A is open if and only if $\mathrm{modid}_{X,A}$ is a continuous map from X into the X modified w.r.t. A.

Let X be a non empty topological space and let X_0 be a subspace of X. The X modified w.r.t. X_0 yielding a strict topological space is defined by the condition (Def. 10).

(Def. 10) Let A be a subset of X. Suppose A = the carrier of X_0. Then the X modified w.r.t. X_0 = the X modified w.r.t. A.

Let X be a non empty topological space and let X_0 be a subspace of X. Note that the X modified w.r.t. X_0 is non empty.

In the sequel X_0 is a non empty subspace of X.

Next we state three propositions:

(114)(i) The carrier of the X modified w.r.t. X_0 = the carrier of X, and

(ii) for every subset A of X such that A = the carrier of X_0 holds the topology of the X modified w.r.t. X_0 = the A-extension of the topology of X.

(115) Let Y_0 be a subspace of the X modified w.r.t. X_0. Suppose the carrier of Y_0 = the carrier of X_0. Then Y_0 is an open subspace of the X modified w.r.t. X_0.

(116) X_0 is an open subspace of X iff the topological structure of X = the X modified w.r.t. X_0.

Let X be a non empty topological space and let X_0 be a subspace of X. The functor modid_{X,X_0} yielding a map from X into the X modified w.r.t. X_0 is defined by:

(Def. 11) For every subset A of X such that A = the carrier of X_0 holds $\mathrm{modid}_{X,X_0} = \mathrm{modid}_{X,A}$.

We now state several propositions:

(117) If X_0 is an open subspace of X, then $\mathrm{modid}_{X,X_0} = \mathrm{id}_X$.

(118) For all non empty subspaces X_0, X_1 of X such that X_0 misses X_1 and for every point x_1 of X_1 holds $\mathrm{modid}_{X,X_0} \restriction X_1$ is continuous at x_1.

(119) For every non empty subspace X_0 of X and for every point x_0 of X_0 holds $\mathrm{modid}_{X,X_0} \restriction X_0$ is continuous at x_0.

(120) Let X_0, X_1 be non empty subspaces of X. Suppose X_0 misses X_1. Then $\mathrm{modid}_{X,X_0} \restriction X_1$ is a continuous map from X_1 into the X modified w.r.t. X_0.

(121) For every non empty subspace X_0 of X holds $\mathrm{modid}_{X,X_0} \restriction X_0$ is a continuous map from X_0 into the X modified w.r.t. X_0.

(122) Let X_0 be a subspace of X. Then X_0 is an open subspace of X if and only if modid_{X,X_0} is a continuous map from X into the X modified w.r.t. X_0.

5. Continuity of Mappings over the Union of Subspaces

In the sequel X, Y are non empty topological spaces.

The following three propositions are true:

(123) Let X_1, X_2 be non empty subspaces of X, g be a map from $X_1 \cup X_2$ into Y, x_1 be a point of X_1, x_2 be a point of X_2, and x be a point of $X_1 \cup X_2$. Suppose $x = x_1$ and $x = x_2$. Then g is continuous at x if and only if $g \restriction X_1$ is continuous at x_1 and $g \restriction X_2$ is continuous at x_2.

(124) Let f be a map from X into Y, X_1, X_2 be non empty subspaces of X, x be a point of $X_1 \cup X_2$, x_1 be a point of X_1, and x_2 be a point of X_2. Suppose $x = x_1$ and $x = x_2$. Then $f \restriction (X_1 \cup X_2)$ is continuous at x if and only if $f \restriction X_1$ is continuous at x_1 and $f \restriction X_2$ is continuous at x_2.

(125) Let f be a map from X into Y and X_1, X_2 be non empty subspaces of X. Suppose $X = X_1 \cup X_2$. Let x be a point of X, x_1 be a point of X_1, and x_2 be a point of X_2. Suppose $x = x_1$ and $x = x_2$. Then f is continuous at x if and only if $f \restriction X_1$ is continuous at x_1 and $f \restriction X_2$ is continuous at x_2.

In the sequel X_1, X_2 are non empty subspaces of X.

We now state a number of propositions:

(126) Suppose X_1 and X_2 are weakly separated. Let g be a map from $X_1 \cup X_2$ into Y. Then g is a continuous map from $X_1 \cup X_2$ into Y if and only if $g \restriction X_1$ is a continuous map from X_1 into Y and $g \restriction X_2$ is a continuous map from X_2 into Y.

(127) Let X_1, X_2 be closed non empty subspaces of X and g be a map from $X_1 \cup X_2$ into Y. Then g is a continuous map from $X_1 \cup X_2$ into Y if and only if $g \restriction X_1$ is a continuous map from X_1 into Y and $g \restriction X_2$ is a continuous map from X_2 into Y.

(128) Let X_1, X_2 be open non empty subspaces of X and g be a map from $X_1 \cup X_2$ into Y. Then g is a continuous map from $X_1 \cup X_2$ into Y if and only if $g \restriction X_1$ is a continuous map from X_1 into Y and $g \restriction X_2$ is a continuous map from X_2 into Y.

(129) Suppose X_1 and X_2 are weakly separated. Let f be a map from X into Y. Then $f \restriction (X_1 \cup X_2)$ is a continuous map from $X_1 \cup X_2$ into Y if and only if $f \restriction X_1$ is a continuous map from X_1 into Y and $f \restriction X_2$ is a continuous map from X_2 into Y.

(130) Let f be a map from X into Y and X_1, X_2 be closed non empty subspaces of X. Then $f \restriction (X_1 \cup X_2)$ is a continuous map from $X_1 \cup X_2$ into Y if and only if $f \restriction X_1$ is a continuous map from X_1 into Y and $f \restriction X_2$ is a continuous map from X_2 into Y.

(131) Let f be a map from X into Y and X_1, X_2 be open non empty subspaces of X. Then $f \restriction (X_1 \cup X_2)$ is a continuous map from $X_1 \cup X_2$ into Y if and only if $f \restriction X_1$ is a continuous map from X_1 into Y and $f \restriction X_2$ is a continuous map from X_2 into Y.

(132) Let f be a map from X into Y and X_1, X_2 be non empty subspaces of X. Suppose $X = X_1 \cup X_2$ and X_1 and X_2 are weakly separated. Then f is a continuous map from X into Y if and only if $f \restriction X_1$ is a continuous map from X_1 into Y and $f \restriction X_2$ is a continuous map from X_2 into Y.

(133) Let f be a map from X into Y and X_1, X_2 be closed non empty subspaces of X. Suppose $X = X_1 \cup X_2$. Then f is a continuous map from X into Y if and only if $f \restriction X_1$ is a continuous map from X_1 into Y and $f \restriction X_2$ is a continuous map from X_2 into Y.

(134) Let f be a map from X into Y and X_1, X_2 be open non empty subspaces of X. Suppose $X = X_1 \cup X_2$. Then f is a continuous map from X into Y if and only if $f \restriction X_1$ is a continuous map from X_1 into Y and $f \restriction X_2$ is a continuous map from X_2 into Y.

(135) X_1 and X_2 are separated if and only if the following conditions are satisfied:

(i) X_1 misses X_2, and

(ii) for every non empty topological space Y and for every map g from $X_1 \cup X_2$ into Y such that $g \restriction X_1$ is a continuous map from X_1 into Y and $g \restriction X_2$ is a continuous map from X_2 into Y holds g is a continuous map from $X_1 \cup X_2$ into Y.

(136) X_1 and X_2 are separated if and only if the following conditions are satisfied:

(i) X_1 misses X_2, and

(ii) for every non empty topological space Y and for every map f from X into Y such that $f \restriction X_1$ is a continuous map from X_1 into Y and $f \restriction X_2$ is a continuous map from X_2 into Y holds $f \restriction (X_1 \cup X_2)$ is a continuous map from $X_1 \cup X_2$ into Y.

(137) Let X_1, X_2 be non empty subspaces of X. Suppose $X = X_1 \cup X_2$. Then X_1 and X_2 are separated if and only if the following conditions are satisfied:

(i) X_1 misses X_2, and

(ii) for every non empty topological space Y and for every map f from X into Y such that $f \restriction X_1$ is a continuous map from X_1 into Y and $f \restriction X_2$ is a continuous map from X_2 into Y holds f is a continuous map from X into Y.

6. The Union of Continuous Mappings

Let X, Y be non empty topological spaces, let X_1, X_2 be non empty subspaces of X, let f_1 be a map from X_1 into Y, and let f_2 be a map from X_2 into Y. Let us assume that X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$. The functor $f_1 \cup f_2$ yielding a map from $X_1 \cup X_2$ into Y is defined as follows:

(Def. 12) $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$.

In the sequel X, Y denote non empty topological spaces.

We now state a number of propositions:

(138) For all non empty subspaces X_1, X_2 of X and for every map g from $X_1 \cup X_2$ into Y holds $g = g \restriction X_1 \cup g \restriction X_2$.

(139) For all non empty subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ and for every map g from X into Y holds $g = g \restriction X_1 \cup g \restriction X_2$.

(140) Let X_1, X_2 be non empty subspaces of X. Suppose X_1 meets X_2. Let f_1 be a map from X_1 into Y and f_2 be a map from X_2 into Y. Then $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$ if and only if $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$.

(141) Let X_1, X_2 be non empty subspaces of X, f_1 be a map from X_1 into Y, and f_2 be a map from X_2 into Y such that $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$. Then

(i) X_1 is a subspace of X_2 iff $f_1 \cup f_2 = f_2$, and

(ii) X_2 is a subspace of X_1 iff $f_1 \cup f_2 = f_1$.

(142) Let X_1, X_2 be non empty subspaces of X, f_1 be a map from X_1 into Y, and f_2 be a map from X_2 into Y. If X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$, then $f_1 \cup f_2 = f_2 \cup f_1$.

(143) Let X_1, X_2, X_3 be non empty subspaces of X, f_1 be a map from X_1 into Y, f_2 be a map from X_2 into Y, and f_3 be a map from X_3 into Y. Suppose X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ but X_1 misses X_3 or $f_1 \restriction (X_1 \cap X_3) = f_3 \restriction (X_1 \cap X_3)$ but X_2 misses X_3 or $f_2 \restriction (X_2 \cap X_3) = f_3 \restriction (X_2 \cap X_3)$. Then $(f_1 \cup f_2) \cup f_3 = f_1 \cup (f_2 \cup f_3)$.

(144) Let X_1, X_2 be non empty subspaces of X. Suppose X_1 meets X_2. Let f_1 be a continuous map from X_1 into Y and f_2 be a continuous map from X_2 into Y. Suppose $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$. Suppose X_1 and X_2 are weakly separated. Then $f_1 \cup f_2$ is a continuous map from $X_1 \cup X_2$ into Y.

(145) Let X_1, X_2 be non empty subspaces of X. Suppose X_1 misses X_2. Let f_1 be a continuous map from X_1 into Y and f_2 be a continuous map from X_2 into Y. Suppose X_1 and X_2 are weakly separated. Then $f_1 \cup f_2$ is a continuous map from $X_1 \cup X_2$ into Y.

(146) Let X_1, X_2 be closed non empty subspaces of X. Suppose X_1 meets X_2. Let f_1 be a continuous map from X_1 into Y and f_2 be a continuous map from X_2 into Y. If $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$, then $f_1 \cup f_2$ is a continuous map from $X_1 \cup X_2$ into Y.

(147) Let X_1, X_2 be open non empty subspaces of X. Suppose X_1 meets X_2. Let f_1 be a continuous map from X_1 into Y and f_2 be a continuous map from X_2 into Y. If $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$, then $f_1 \cup f_2$ is a continuous map from $X_1 \cup X_2$ into Y.

(148) Let X_1, X_2 be closed non empty subspaces of X. Suppose X_1 misses X_2. Let f_1 be a continuous map from X_1 into Y and f_2 be a continuous map from X_2 into Y. Then $f_1 \cup f_2$ is a continuous map from $X_1 \cup X_2$ into Y.

(149) Let X_1, X_2 be open non empty subspaces of X. Suppose X_1 misses X_2. Let f_1 be a continuous map from X_1 into Y and f_2 be a continuous map from X_2 into Y. Then $f_1 \cup f_2$ is a continuous map from $X_1 \cup X_2$ into Y.

(150) Let X_1, X_2 be non empty subspaces of X. Then X_1 and X_2 are separated if and only if the following conditions are satisfied:

(i) X_1 misses X_2, and

(ii) for every non empty topological space Y and for every continuous map f_1 from X_1 into Y and for every continuous map f_2 from X_2 into Y holds $f_1 \cup f_2$ is a continuous map from $X_1 \cup X_2$ into Y.

(151) Let X_1, X_2 be non empty subspaces of X. Suppose $X = X_1 \cup X_2$. Let f_1 be a continuous map from X_1 into Y and f_2 be a continuous map from X_2 into Y. Suppose $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$. Suppose X_1 and X_2 are weakly separated. Then $f_1 \cup f_2$ is a continuous map from X into Y.

(152) Let X_1, X_2 be closed non empty subspaces of X, f_1 be a continuous map from X_1 into Y, and f_2 be a continuous map from X_2 into Y. Suppose $X = X_1 \cup X_2$ and $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$. Then $f_1 \cup f_2$ is a continuous map from X into Y.

(153) Let X_1, X_2 be open non empty subspaces of X, f_1 be a continuous map from X_1 into Y, and f_2 be a continuous map from X_2 into Y. Suppose $X = X_1 \cup X_2$ and $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$. Then $f_1 \cup f_2$ is a continuous map from X into Y.

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