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On the Rectangular Finite Sequences of the Points of the Plane

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Summary. The article deals with a rather technical concept – rectangular sequences of the points of the plane. We mean by that a finite sequence consisting of five elements, that is circular, i.e. the first element and the fifth one of it are equal, and such that the polygon determined by it is a non degenerated rectangle, with sides parallel to axes. The main result is that for the rectangle determined by such a sequence the left and the right component of the complement of it are different and disjoint.

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The articles [23], [7], [29], [30], [2], [26], [13], [1], [27], [5], [6], [3], [28], [25], [16], [15], [14], [4], [22], [21], [10], [20], [11], [12], [18], [19], [24], [17], [8], and [9] provide the notation and terminology for this paper.

1. General preliminaries

The following proposition is true

(1) For every trivial set A and for every set B such that $B \subseteq A$ holds B is trivial.

One can check that every function which is non constant is also non trivial.
Let us mention that every function which is trivial is also constant.
Next we state the proposition

(2) For every function f such that rng f is trivial holds f is constant.

Let f be a constant function. Note that rng f is trivial.
Let us note that there exists a finite sequence which is non empty and constant.
We now state three propositions:

(3) For all finite sequences f, g such that $f \frown g$ is constant holds f is constant and g is constant.

(4) For all sets x, y such that $\langle x, y \rangle$ is constant holds $x = y$.

(5) For all sets x, y, z such that $\langle x, y, z \rangle$ is constant holds $x = y$ and $y = z$ and $z = x$.

2. PRELIMINARIES (GENERAL TOPOLOGY)

Next we state four propositions:

(6) Let G_1 be a non empty topological space, A be a subset of G_1, and B be a non empty subset of G_1. If A is a component of B, then $A \neq \emptyset$.

(7) For every topological structure G_1 and for all subsets A, B of G_1 such that A is a component of B holds $A \subseteq B$.

(8) Let T be a non empty topological space, A be a non empty subset of T, and B_1, B_2, S be subsets of T. Suppose B_1 is a component of A and B_2 is a component of A and S is a component of A and $B_1 \cup B_2 = A$. Then $S = B_1$ or $S = B_2$.

(9) Let T be a non empty topological space, A be a non empty subset of T, and B_1, B_2, C_1, C_2 be subsets of T. Suppose B_1 is a component of A and B_2 is a component of A and C_1 is a component of A and C_2 is a component of A and $B_1 \cup B_2 = A$ and $C_1 \cup C_2 = A$. Then $\{B_1, B_2\} = \{C_1, C_2\}$.

3. PRELIMINARIES (THE TOPOLOGY OF THE PLANE)

We adopt the following rules: S denotes a subset of $\mathcal{E}^2_{\mathrm{T}}$, C, C_1, C_2 denote non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and p, q denote points of $\mathcal{E}^2_{\mathrm{T}}$.

One can prove the following proposition

(10) For all points p, q, r of $\mathcal{E}^2_{\mathrm{T}}$ holds $\widetilde{\mathcal{L}}(\langle p, q, r \rangle) = \mathcal{L}(p, q) \cup \mathcal{L}(q, r)$.

Let n be a natural number and let f be a non trivial finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$. Observe that $\widetilde{\mathcal{L}}(f)$ is non empty.

Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Note that $\widetilde{\mathcal{L}}(f)$ is compact.

We now state two propositions:

(11) For all subsets A, B of $\mathcal{E}^2_{\mathrm{T}}$ such that $A \subseteq B$ and B is horizontal holds A is horizontal.

(12) For all subsets A, B of $\mathcal{E}^2_{\mathrm{T}}$ such that $A \subseteq B$ and B is vertical holds A is vertical.

Let us observe that $\square_{\mathcal{E}^2}$ is special polygonal, non horizontal, and non vertical.

One can verify that there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is non vertical, non horizontal, non empty, and compact.

4. SPECIAL POINTS OF A COMPACT NON EMPTY SUBSET OF THE PLANE

One can prove the following propositions:

(13) $\mathrm{N}_{\min}(C) \in C$ and $\mathrm{N}_{\max}(C) \in C$.

(14) $\mathrm{S}_{\min}(C) \in C$ and $\mathrm{S}_{\max}(C) \in C$.

(15) $\mathrm{W}_{\min}(C) \in C$ and $\mathrm{W}_{\max}(C) \in C$.

(16) $\mathrm{E}_{\min}(C) \in C$ and $\mathrm{E}_{\max}(C) \in C$.

(17) C is vertical iff W-bound$(C) =$ E-bound(C).

(18) C is horizontal iff S-bound$(C) =$ N-bound(C).

(19) If NW-corner$(C) =$ NE-corner(C), then C is vertical.

(20) If SW-corner$(C) =$ SE-corner(C), then C is vertical.

(21) If NW-corner$(C) =$ SW-corner(C), then C is horizontal.

(22) If NE-corner(C) = SE-corner(C), then C is horizontal.

In the sequel t, r_1, r_2, s_1, s_2 are real numbers.
Next we state a number of propositions:

(23) W-bound$(C) \leq$ E-bound(C).

(24) S-bound$(C) \leq$ N-bound(C).

(25) $\mathcal{L}(\text{SE-corner}(C), \text{NE-corner}(C)) = \{p : p_{\mathbf{1}} = \text{E-bound}(C) \ \wedge \ p_{\mathbf{2}} \leq \text{N-bound}(C) \ \wedge \ p_{\mathbf{2}} \geq \text{S-bound}(C)\}$.

(26) $\mathcal{L}(\text{SW-corner}(C), \text{SE-corner}(C)) = \{p : p_{\mathbf{1}} \leq \text{E-bound}(C) \ \wedge \ p_{\mathbf{1}} \geq \text{W-bound}(C) \ \wedge \ p_{\mathbf{2}} = \text{S-bound}(C)\}$.

(27) $\mathcal{L}(\text{NW-corner}(C), \text{NE-corner}(C)) = \{p : p_{\mathbf{1}} \leq \text{E-bound}(C) \ \wedge \ p_{\mathbf{1}} \geq \text{W-bound}(C) \ \wedge \ p_{\mathbf{2}} = \text{N-bound}(C)\}$.

(28) $\mathcal{L}(\text{SW-corner}(C), \text{NW-corner}(C)) = \{p : p_{\mathbf{1}} = \text{W-bound}(C) \ \wedge \ p_{\mathbf{2}} \leq \text{N-bound}(C) \ \wedge \ p_{\mathbf{2}} \geq \text{S-bound}(C)\}$.

(29) $\mathcal{L}(\text{SW-corner}(C), \text{NW-corner}(C)) \cap \mathcal{L}(\text{NW-corner}(C), \text{NE-corner}(C)) = \{\text{NW-corner}(C)\}$.

(30) $\mathcal{L}(\text{NW-corner}(C), \text{NE-corner}(C)) \cap \mathcal{L}(\text{NE-corner}(C), \text{SE-corner}(C)) = \{\text{NE-corner}(C)\}$.

(31) $\mathcal{L}(\text{SE-corner}(C), \text{NE-corner}(C)) \cap \mathcal{L}(\text{SW-corner}(C), \text{SE-corner}(C)) = \{\text{SE-corner}(C)\}$.

(32) $\mathcal{L}(\text{NW-corner}(C), \text{SW-corner}(C)) \cap \mathcal{L}(\text{SW-corner}(C), \text{SE-corner}(C)) = \{\text{SW-corner}(C)\}$.

5. Subsets of the Plane that are Neither Vertical nor Horizontal

In the sequel D_1 denotes a non vertical non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, D_2 denotes a non horizontal non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, and D denotes a non vertical non horizontal non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

Next we state four propositions:

(33) W-bound$(D_1) <$ E-bound(D_1).

(34) S-bound$(D_2) <$ N-bound(D_2).

(35) $\mathcal{L}(\text{SW-corner}(D_1), \text{NW-corner}(D_1))$ misses $\mathcal{L}(\text{SE-corner}(D_1), \text{NE-corner}(D_1))$.

(36) $\mathcal{L}(\text{SW-corner}(D_2), \text{SE-corner}(D_2))$ misses $\mathcal{L}(\text{NW-corner}(D_2), \text{NE-corner}(D_2))$.

6. A Special Sequence Related to a Compact Non Empty Subset of the Plane

Let C be a subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor SpStSeq$C$ yields a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 1) $\text{SpStSeq}\,C = \langle \text{NW-corner}(C), \text{NE-corner}(C), \text{SE-corner}(C) \rangle \frown \langle \text{SW-corner}(C), \text{NW-corner}(C) \rangle$.

The following propositions are true:

(37) $(\text{SpStSeq}\,S)_1 = \text{NW-corner}(S)$.

(38) $(\text{SpStSeq}\,S)_2 = \text{NE-corner}(S)$.

(39) $(\text{SpStSeq}\,S)_3 = \text{SE-corner}(S)$.

(40) $(\text{SpStSeq}\,S)_4 = \text{SW-corner}(S)$.

(41) $(\text{SpStSeq}\,S)_5 = \text{NW-corner}(S)$.

(42) $\operatorname{len}\operatorname{SpStSeq}S = 5$.

(43) $\widetilde{\mathcal{L}}(\operatorname{SpStSeq}S) = \mathcal{L}(\text{NW-corner}(S), \text{NE-corner}(S)) \cup \mathcal{L}(\text{NE-corner}(S), \text{SE-corner}(S)) \cup (\mathcal{L}(\text{SE-corner}(S), \text{SW-corner}($ $\mathcal{L}(\text{SW-corner}(S), \text{NW-corner}(S)))$.

Let D be a non vertical non empty compact subset of $\mathcal{E}^2_{\rm T}$. One can check that $\operatorname{SpStSeq}D$ is non constant.

Let D be a non horizontal non empty compact subset of $\mathcal{E}^2_{\rm T}$. Note that $\operatorname{SpStSeq}D$ is non constant.

Let D be a non vertical non horizontal non empty compact subset of $\mathcal{E}^2_{\rm T}$. Observe that $\operatorname{SpStSeq}D$ is special, unfolded, circular, s.c.c., and standard.

One can prove the following propositions:

(44) $\widetilde{\mathcal{L}}(\operatorname{SpStSeq}D) = [\text{W-bound}(D), \text{E-bound}(D), \text{S-bound}(D), \text{N-bound}(D)]$.

(45) For every non empty topological structure T and for every subset X of T and for every real map f of T holds $\operatorname{rng}(f \restriction X) = f^\circ X$.

(46) Let T be a non empty topological space, X be a non empty compact subset of T, and f be a continuous real map of T. Then $f^\circ X$ is lower bounded.

(47) Let T be a non empty topological space, X be a non empty compact subset of T, and f be a continuous real map of T. Then $f^\circ X$ is upper bounded.

One can verify that there exists a subset of $\mathbb{R}$ which is non empty, upper bounded, and lower bounded.

We now state a number of propositions:

(48) $\text{W-bound}(S) = \inf(\text{proj1}^\circ S)$.

(49) $\text{S-bound}(S) = \inf(\text{proj2}^\circ S)$.

(50) $\text{N-bound}(S) = \sup(\text{proj2}^\circ S)$.

(51) $\text{E-bound}(S) = \sup(\text{proj1}^\circ S)$.

(52) For all non empty lower bounded subsets A, B of $\mathbb{R}$ holds $\inf(A \cup B) = \min(\inf A, \inf B)$.

(53) For all non empty upper bounded subsets A, B of $\mathbb{R}$ holds $\sup(A \cup B) = \max(\sup A, \sup B)$.

(54) If $S = C_1 \cup C_2$, then $\text{W-bound}(S) = \min(\text{W-bound}(C_1), \text{W-bound}(C_2))$.

(55) If $S = C_1 \cup C_2$, then $\text{S-bound}(S) = \min(\text{S-bound}(C_1), \text{S-bound}(C_2))$.

(56) If $S = C_1 \cup C_2$, then $\text{N-bound}(S) = \max(\text{N-bound}(C_1), \text{N-bound}(C_2))$.

(57) If $S = C_1 \cup C_2$, then $\text{E-bound}(S) = \max(\text{E-bound}(C_1), \text{E-bound}(C_2))$.

Let us consider p, q. Note that $\mathcal{L}(p, q)$ is compact.

Let us observe that $\emptyset_{\mathbb{R}}$ is bounded.

Let us consider r_1, r_2. Note that $[r_1, r_2]$ is bounded.

Let us observe that every subset of $\mathbb{R}$ which is bounded is also lower bounded and upper bounded and every subset of $\mathbb{R}$ which is lower bounded and upper bounded is also bounded.

One can prove the following propositions:

(59)[1] If $r_1 \leq r_2$, then $t \in [r_1, r_2]$ iff there exists s_1 such that $0 \leq s_1$ and $s_1 \leq 1$ and $t = s_1 \cdot r_1 + (1 - s_1) \cdot r_2$.

(60) If $p_{\mathbf{1}} \leq q_{\mathbf{1}}$, then $\text{proj1}^\circ \mathcal{L}(p, q) = [p_{\mathbf{1}}, q_{\mathbf{1}}]$.

(61) If $p_{\mathbf{2}} \leq q_{\mathbf{2}}$, then $\text{proj2}^\circ \mathcal{L}(p, q) = [p_{\mathbf{2}}, q_{\mathbf{2}}]$.

[1] The proposition (58) has been removed.

(62) If $p_{\mathbf{1}} \leq q_{\mathbf{1}}$, then W-bound$(\mathcal{L}(p,q)) = p_{\mathbf{1}}$.

(63) If $p_{\mathbf{2}} \leq q_{\mathbf{2}}$, then S-bound$(\mathcal{L}(p,q)) = p_{\mathbf{2}}$.

(64) If $p_{\mathbf{2}} \leq q_{\mathbf{2}}$, then N-bound$(\mathcal{L}(p,q)) = q_{\mathbf{2}}$.

(65) If $p_{\mathbf{1}} \leq q_{\mathbf{1}}$, then E-bound$(\mathcal{L}(p,q)) = q_{\mathbf{1}}$.

(66) W-bound$(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) =$ W-bound(C).

(67) S-bound$(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) =$ S-bound(C).

(68) N-bound$(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) =$ N-bound(C).

(69) E-bound$(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) =$ E-bound(C).

(70) NW-corner$(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) =$ NW-corner(C).

(71) NE-corner$(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) =$ NE-corner(C).

(72) SW-corner$(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) =$ SW-corner(C).

(73) SE-corner$(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) =$ SE-corner(C).

(74) $\text{W}_{\text{most}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \mathcal{L}(\text{SW-corner}(C), \text{NW-corner}(C))$.

(75) $\text{N}_{\text{most}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \mathcal{L}(\text{NW-corner}(C), \text{NE-corner}(C))$.

(76) $\text{S}_{\text{most}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \mathcal{L}(\text{SW-corner}(C), \text{SE-corner}(C))$.

(77) $\text{E}_{\text{most}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \mathcal{L}(\text{SE-corner}(C), \text{NE-corner}(C))$.

(78) $\text{proj2}^{\circ}\, \mathcal{L}(\text{SW-corner}(C), \text{NW-corner}(C)) = [\text{S-bound}(C), \text{N-bound}(C)]$.

(79) $\text{proj1}^{\circ}\, \mathcal{L}(\text{NW-corner}(C), \text{NE-corner}(C)) = [\text{W-bound}(C), \text{E-bound}(C)]$.

(80) $\text{proj2}^{\circ}\, \mathcal{L}(\text{NE-corner}(C), \text{SE-corner}(C)) = [\text{S-bound}(C), \text{N-bound}(C)]$.

(81) $\text{proj1}^{\circ}\, \mathcal{L}(\text{SE-corner}(C), \text{SW-corner}(C)) = [\text{W-bound}(C), \text{E-bound}(C)]$.

(82) $\text{W}_{\text{min}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \text{SW-corner}(C)$.

(83) $\text{W}_{\text{max}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \text{NW-corner}(C)$.

(84) $\text{N}_{\text{min}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \text{NW-corner}(C)$.

(85) $\text{N}_{\text{max}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \text{NE-corner}(C)$.

(86) $\text{E}_{\text{min}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \text{SE-corner}(C)$.

(87) $\text{E}_{\text{max}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \text{NE-corner}(C)$.

(88) $\text{S}_{\text{min}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \text{SW-corner}(C)$.

(89) $\text{S}_{\text{max}}(\widetilde{\mathcal{L}}(\text{SpStSeq}\,C)) = \text{SE-corner}(C)$.

7. Rectangular finite suequences of the points of the plane

Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. We say that f is rectangular if and only if:

(Def. 2) There exists D such that $f = \mathrm{SpStSeq}\,D$.

Let us consider D. Observe that $\mathrm{SpStSeq}\,D$ is rectangular.
One can check that there exists a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ which is rectangular.
In the sequel s denotes a rectangular finite sequence of elements of $\mathcal{E}^2_{\rm T}$.
The following proposition is true

(90) $\mathrm{len}\,s = 5$.

Let us observe that every finite sequence of elements of $\mathcal{E}^2_{\rm T}$ which is rectangular is also non constant.

Let us note that every non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ which is rectangular is also standard, special, unfolded, circular, and s.c.c..

One can prove the following four propositions:

(91) $s_1 = \mathrm{N}_{\rm min}(\widetilde{\mathcal{L}}(s))$ and $s_1 = \mathrm{W}_{\rm max}(\widetilde{\mathcal{L}}(s))$.

(92) $s_2 = \mathrm{N}_{\rm max}(\widetilde{\mathcal{L}}(s))$ and $s_2 = \mathrm{E}_{\rm max}(\widetilde{\mathcal{L}}(s))$.

(93) $s_3 = \mathrm{S}_{\rm max}(\widetilde{\mathcal{L}}(s))$ and $s_3 = \mathrm{E}_{\rm min}(\widetilde{\mathcal{L}}(s))$.

(94) $s_4 = \mathrm{S}_{\rm min}(\widetilde{\mathcal{L}}(s))$ and $s_4 = \mathrm{W}_{\rm min}(\widetilde{\mathcal{L}}(s))$.

8. Jordan property

The following proposition is true

(95) If $r_1 < r_2$ and $s_1 < s_2$, then $[r_1, r_2, s_1, s_2]$ is Jordan.

Let f be a rectangular finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Observe that $\widetilde{\mathcal{L}}(f)$ is Jordan.

Let S be a subset of $\mathcal{E}^2_{\rm T}$. Let us observe that S is Jordan if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $S^{\rm c} \neq \emptyset$, and

(ii) there exist subsets A_1, A_2 of $\mathcal{E}^2_{\rm T}$ such that $S^{\rm c} = A_1 \cup A_2$ and A_1 misses A_2 and $\overline{A_1} \setminus A_1 = \overline{A_2} \setminus A_2$ and A_1 is a component of $S^{\rm c}$ and A_2 is a component of $S^{\rm c}$.

Next we state the proposition

(96) For every rectangular finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{LeftComp}(f)$ misses $\mathrm{RightComp}(f)$.

Let f be a non constant standard special circular sequence. One can check that $\mathrm{LeftComp}(f)$ is non empty and $\mathrm{RightComp}(f)$ is non empty.

One can prove the following proposition

(97) For every rectangular finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{LeftComp}(f) \neq \mathrm{RightComp}(f)$.

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