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Trigonometric Functions on Complex Space

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Summary. This article describes definitions of sine, cosine, hyperbolic sine and hyperbolic cosine. Some of their basic properties are discussed.

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The articles [11], [9], [5], [12], [1], [10], [4], [3], [2], [6], [8], [13], and [7] provide the notation and terminology for this paper.

1. Definitions of Trigonometric Functions

We use the following convention: x, y are elements of $\mathbb{R}$, z, z_1, z_2 are elements of $\mathbb{C}$, and n is a natural number.

The function $\sin_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 1) $\sin_{\mathbb{C}}(z) = \frac{\exp(i\cdot z)-\exp(-i\cdot z)}{(2+0i)\cdot i}$.

The function $\cos_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined as follows:

(Def. 2) $\cos_{\mathbb{C}}(z) = \frac{\exp(i\cdot z)+\exp(-i\cdot z)}{2+0i}$.

The function $\sinh_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 3) $\sinh_{\mathbb{C}}(z) = \frac{\exp z-\exp(-z)}{2+0i}$.

The function $\cosh_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 4) $\cosh_{\mathbb{C}}(z) = \frac{\exp z+\exp(-z)}{2+0i}$.

2. Properties of Trigonometric Functions on Complex Space

One can prove the following propositions:

(1) For every element z of $\mathbb{C}$ holds $\sin_{\mathbb{C}z}\cdot\sin_{\mathbb{C}z}+\cos_{\mathbb{C}z}\cdot\cos_{\mathbb{C}z} = 1_{\mathbb{C}}$.

(2) $-\sin_{\mathbb{C}z} = \sin_{\mathbb{C}-z}$.

(3) $\cos_{\mathbb{C}z} = \cos_{\mathbb{C}-z}$.

(4) $\sin_{\mathbb{C}}{}_{z_1+z_2} = \sin_{\mathbb{C}}{}_{z_1} \cdot \cos_{\mathbb{C}}{}_{z_2} + \cos_{\mathbb{C}}{}_{z_1} \cdot \sin_{\mathbb{C}}{}_{z_2}$.

(5) $\sin_{\mathbb{C}}{}_{z_1-z_2} = \sin_{\mathbb{C}}{}_{z_1} \cdot \cos_{\mathbb{C}}{}_{z_2} - \cos_{\mathbb{C}}{}_{z_1} \cdot \sin_{\mathbb{C}}{}_{z_2}$.

(6) $\cos_{\mathbb{C}}{}_{z_1+z_2} = \cos_{\mathbb{C}}{}_{z_1} \cdot \cos_{\mathbb{C}}{}_{z_2} - \sin_{\mathbb{C}}{}_{z_1} \cdot \sin_{\mathbb{C}}{}_{z_2}$.

(7) $\cos_{\mathbb{C}}{}_{z_1-z_2} = \cos_{\mathbb{C}}{}_{z_1} \cdot \cos_{\mathbb{C}}{}_{z_2} + \sin_{\mathbb{C}}{}_{z_1} \cdot \sin_{\mathbb{C}}{}_{z_2}$.

(8) $\cosh_{\mathbb{C}}{}_{z} \cdot \cosh_{\mathbb{C}}{}_{z} - \sinh_{\mathbb{C}}{}_{z} \cdot \sinh_{\mathbb{C}}{}_{z} = 1_{\mathbb{C}}$.

(9) $-\sinh_{\mathbb{C}}{}_{z} = \sinh_{\mathbb{C}}{}_{-z}$.

(10) $\cosh_{\mathbb{C}}{}_{z} = \cosh_{\mathbb{C}}{}_{-z}$.

(11) $\sinh_{\mathbb{C}}{}_{z_1+z_2} = \sinh_{\mathbb{C}}{}_{z_1} \cdot \cosh_{\mathbb{C}}{}_{z_2} + \cosh_{\mathbb{C}}{}_{z_1} \cdot \sinh_{\mathbb{C}}{}_{z_2}$.

(12) $\sinh_{\mathbb{C}}{}_{z_1-z_2} = \sinh_{\mathbb{C}}{}_{z_1} \cdot \cosh_{\mathbb{C}}{}_{z_2} - \cosh_{\mathbb{C}}{}_{z_1} \cdot \sinh_{\mathbb{C}}{}_{z_2}$.

(13) $\cosh_{\mathbb{C}}{}_{z_1-z_2} = \cosh_{\mathbb{C}}{}_{z_1} \cdot \cosh_{\mathbb{C}}{}_{z_2} - \sinh_{\mathbb{C}}{}_{z_1} \cdot \sinh_{\mathbb{C}}{}_{z_2}$.

(14) $\cosh_{\mathbb{C}}{}_{z_1+z_2} = \cosh_{\mathbb{C}}{}_{z_1} \cdot \cosh_{\mathbb{C}}{}_{z_2} + \sinh_{\mathbb{C}}{}_{z_1} \cdot \sinh_{\mathbb{C}}{}_{z_2}$.

(15) $\sin_{\mathbb{C}}{}_{i \cdot z} = i \cdot \sinh_{\mathbb{C}}{}_{z}$.

(16) $\cos_{\mathbb{C}}{}_{i \cdot z} = \cosh_{\mathbb{C}}{}_{z}$.

(17) $\sinh_{\mathbb{C}}{}_{i \cdot z} = i \cdot \sin_{\mathbb{C}}{}_{z}$.

(18) $\cosh_{\mathbb{C}}{}_{i \cdot z} = \cos_{\mathbb{C}}{}_{z}$.

(19) For all elements x, y of $\mathbb{R}$ holds $\exp(x+yi) = \exp(x) \cdot \cos(y) + (\exp(x) \cdot \sin(y))i$.

(20) $\exp(0_{\mathbb{C}}) = 1+0i$.

(21) $\sin_{\mathbb{C}}{}_{0_{\mathbb{C}}} = 0_{\mathbb{C}}$.

(22) $\sinh_{\mathbb{C}}{}_{0_{\mathbb{C}}} = 0_{\mathbb{C}}$.

(23) $\cos_{\mathbb{C}}{}_{0_{\mathbb{C}}} = 1+0i$.

(24) $\cosh_{\mathbb{C}}{}_{0_{\mathbb{C}}} = 1+0i$.

(25) $\exp z = \cosh_{\mathbb{C}}{}_{z} + \sinh_{\mathbb{C}}{}_{z}$.

(26) $\exp(-z) = \cosh_{\mathbb{C}}{}_{z} - \sinh_{\mathbb{C}}{}_{z}$.

(27) $\exp(z+(2 \cdot \pi+0i) \cdot i) = \exp z$ and $\exp(z+(0+(2 \cdot \pi)i)) = \exp z$.

(28) $\exp(0+(2 \cdot \pi \cdot n)i) = 1+0i$ and $\exp((2 \cdot \pi \cdot n+0i) \cdot i) = 1+0i$.

(29) $\exp(0+(-2 \cdot \pi \cdot n)i) = 1+0i$ and $\exp((-2 \cdot \pi \cdot n+0i) \cdot i) = 1+0i$.

(30) $\exp(0+((2 \cdot n+1) \cdot \pi)i) = -1+0i$ and $\exp(((2 \cdot n+1) \cdot \pi+0i) \cdot i) = -1+0i$.

(31) $\exp(0+(-(2 \cdot n+1) \cdot \pi)i) = -1+0i$ and $\exp((-(2 \cdot n+1) \cdot \pi+0i) \cdot i) = -1+0i$.

(32) $\exp(0+((2 \cdot n+\frac{1}{2}) \cdot \pi)i) = 0+1i$ and $\exp(((2 \cdot n+\frac{1}{2}) \cdot \pi+0i) \cdot i) = 0+1i$.

(33) $\exp(0+(-(2 \cdot n+\frac{1}{2}) \cdot \pi)i) = 0+(-1)i$ and $\exp((-(2 \cdot n+\frac{1}{2}) \cdot \pi+0i) \cdot i) = 0+(-1)i$.

(34) $\sin_{\mathbb{C}}{}_{z+(2 \cdot n \cdot \pi+0i)} = \sin_{\mathbb{C}}{}_{z}$.

(35) $\cos_{\mathbb{C}}{}_{z+(2 \cdot n \cdot \pi+0i)} = \cos_{\mathbb{C}}{}_{z}$.

(36) $\exp(i \cdot z) = \cos_{\mathbb{C}}{}_{z} + i \cdot \sin_{\mathbb{C}}{}_{z}$.

(37) $\exp(-i \cdot z) = \cos_{\mathbb{C}}{}_{z} - i \cdot \sin_{\mathbb{C}}{}_{z}$.

(38) For every element x of $\mathbb{R}$ holds $\sin_{\mathbb{C}x+0i} = \sin(x)+0i$.

(39) For every element x of $\mathbb{R}$ holds $\cos_{\mathbb{C}x+0i} = \cos(x)+0i$.

(40) For every element x of $\mathbb{R}$ holds $\sinh_{\mathbb{C}x+0i} = \sinh(x)+0i$.

(41) For every element x of $\mathbb{R}$ holds $\cosh_{\mathbb{C}x+0i} = \cosh(x)+0i$.

(42) For all elements x, y of $\mathbb{R}$ holds $x+yi = (x+0i)+i\cdot(y+0i)$.

(43) $\sin_{\mathbb{C}x+yi} = \sin(x)\cdot\cosh(y)+(\cos(x)\cdot\sinh(y))i$.

(44) $\sin_{\mathbb{C}x+(-y)i} = \sin(x)\cdot\cosh(y)+(-\cos(x)\cdot\sinh(y))i$.

(45) $\cos_{\mathbb{C}x+yi} = \cos(x)\cdot\cosh(y)+(-\sin(x)\cdot\sinh(y))i$.

(46) $\cos_{\mathbb{C}x+(-y)i} = \cos(x)\cdot\cosh(y)+(\sin(x)\cdot\sinh(y))i$.

(47) $\sinh_{\mathbb{C}x+yi} = \sinh(x)\cdot\cos(y)+(\cosh(x)\cdot\sin(y))i$.

(48) $\sinh_{\mathbb{C}x+(-y)i} = \sinh(x)\cdot\cos(y)+(-\cosh(x)\cdot\sin(y))i$.

(49) $\cosh_{\mathbb{C}x+yi} = \cosh(x)\cdot\cos(y)+(\sinh(x)\cdot\sin(y))i$.

(50) $\cosh_{\mathbb{C}x+(-y)i} = \cosh(x)\cdot\cos(y)+(-\sinh(x)\cdot\sin(y))i$.

(51) For every natural number n and for every element z of $\mathbb{C}$ holds $(\cos_{\mathbb{C}z}+i\cdot\sin_{\mathbb{C}z})^{n}_{\mathbb{N}} = \cos_{\mathbb{C}(n+0i)\cdot z}+i\cdot\sin_{\mathbb{C}(n+0i)\cdot z}$.

(52) For every natural number n and for every element z of $\mathbb{C}$ holds $(\cos_{\mathbb{C}z}-i\cdot\sin_{\mathbb{C}z})^{n}_{\mathbb{N}} = \cos_{\mathbb{C}(n+0i)\cdot z}-i\cdot\sin_{\mathbb{C}(n+0i)\cdot z}$.

(53) For every natural number n and for every element z of $\mathbb{C}$ holds $\exp(i\cdot(n+0i)\cdot z) = (\cos_{\mathbb{C}z}+i\cdot\sin_{\mathbb{C}z})^{n}_{\mathbb{N}}$.

(54) For every natural number n and for every element z of $\mathbb{C}$ holds $\exp(-i\cdot(n+0i)\cdot z) = (\cos_{\mathbb{C}z}-i\cdot\sin_{\mathbb{C}z})^{n}_{\mathbb{N}}$.

(55) For all elements x, y of $\mathbb{R}$ holds $\frac{1+(-1)i}{2+0i}\cdot\sinh_{\mathbb{C}x+yi}+\frac{1+1i}{2+0i}\cdot\sinh_{\mathbb{C}x+(-y)i} = (\sinh(x)\cdot\cos(y)+\cosh(x)\cdot\sin(y))+0i$.

(56) For all elements x, y of $\mathbb{R}$ holds $\frac{1+(-1)i}{2+0i}\cdot\cosh_{\mathbb{C}x+yi}+\frac{1+1i}{2+0i}\cdot\cosh_{\mathbb{C}x+(-y)i} = (\sinh(x)\cdot\sin(y)+\cosh(x)\cdot\cos(y))+0i$.

(57) $\sinh_{\mathbb{C}z}\cdot\sinh_{\mathbb{C}z} = \frac{\cosh_{\mathbb{C}(2+0i)\cdot z}-(1+0i)}{2+0i}$.

(58) $\cosh_{\mathbb{C}z}\cdot\cosh_{\mathbb{C}z} = \frac{\cosh_{\mathbb{C}(2+0i)\cdot z}+(1+0i)}{2+0i}$.

(59) $\sinh_{\mathbb{C}(2+0i)\cdot z} = (2+0i)\cdot\sinh_{\mathbb{C}z}\cdot\cosh_{\mathbb{C}z}$ and $\cosh_{\mathbb{C}(2+0i)\cdot z} = (2+0i)\cdot\cosh_{\mathbb{C}z}\cdot\cosh_{\mathbb{C}z}-(1+0i)$.

(60) $\sinh_{\mathbb{C}z_1}\cdot\sinh_{\mathbb{C}z_1}-\sinh_{\mathbb{C}z_2}\cdot\sinh_{\mathbb{C}z_2} = \sinh_{\mathbb{C}z_1+z_2}\cdot\sinh_{\mathbb{C}z_1-z_2}$ and $\cosh_{\mathbb{C}z_1}\cdot\cosh_{\mathbb{C}z_1}-\cosh_{\mathbb{C}z_2}\cdot\cosh_{\mathbb{C}z_2} = \sinh_{\mathbb{C}z_1+z_2}\cdot\sinh_{\mathbb{C}z_1-z_2}$ and $\sinh_{\mathbb{C}z_1}\cdot\sinh_{\mathbb{C}z_1}-\sinh_{\mathbb{C}z_2}\cdot\sinh_{\mathbb{C}z_2} = \cosh_{\mathbb{C}z_1}\cdot\cosh_{\mathbb{C}z_1}-\cosh_{\mathbb{C}z_2}\cdot\cosh_{\mathbb{C}z_2}$.

(61) $\cosh_{\mathbb{C}z_1+z_2}\cdot\cosh_{\mathbb{C}z_1-z_2} = \sinh_{\mathbb{C}z_1}\cdot\sinh_{\mathbb{C}z_1}+\cosh_{\mathbb{C}z_2}\cdot\cosh_{\mathbb{C}z_2}$ and $\cosh_{\mathbb{C}z_1+z_2}\cdot\cosh_{\mathbb{C}z_1-z_2} = \cosh_{\mathbb{C}z_1}\cdot\cosh_{\mathbb{C}z_1}+\sinh_{\mathbb{C}z_2}\cdot\sinh_{\mathbb{C}z_2}$ and $\sinh_{\mathbb{C}z_1}\cdot\sinh_{\mathbb{C}z_1}+\cosh_{\mathbb{C}z_2}\cdot\cosh_{\mathbb{C}z_2} = \cosh_{\mathbb{C}z_1}\cdot\cosh_{\mathbb{C}z_1}+\sinh_{\mathbb{C}z_2}\cdot\sinh_{\mathbb{C}z_2}$.

(62) $\sinh_{\mathbb{C}(2+0i)\cdot z_1}+\sinh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i)\cdot\sinh_{\mathbb{C}z_1+z_2}\cdot\cosh_{\mathbb{C}z_1-z_2}$ and $\sinh_{\mathbb{C}(2+0i)\cdot z_1}-\sinh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i)\cdot\sinh_{\mathbb{C}z_1-z_2}\cdot\cosh_{\mathbb{C}z_1+z_2}$.

(63) $\cosh_{\mathbb{C}(2+0i)\cdot z_1}+\cosh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i)\cdot\cosh_{\mathbb{C}z_1+z_2}\cdot\cosh_{\mathbb{C}z_1-z_2}$ and $\cosh_{\mathbb{C}(2+0i)\cdot z_1}-\cosh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i)\cdot\sinh_{\mathbb{C}z_1+z_2}\cdot\sinh_{\mathbb{C}z_1-z_2}$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[3] Czesław Byliński. The complex numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/complex1.html.

[4] Library Committee. Introduction to arithmetic. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/arytm_0.html.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[6] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[7] Takashi Mitsuishi and Yuguang Yang. Properties of the trigonometric function. *Journal of Formalized Mathematics*, 11, 1999. http://mizar.org/JFM/Vol11/sin_cos2.html.

[8] Yasunari Shidama and Artur Korniłowicz. Convergence and the limit of complex sequences. Series. *Journal of Formalized Mathematics*, 9, 1997. http://mizar.org/JFM/Vol9/comseq_3.html.

[9] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[10] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[11] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[12] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[13] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Journal of Formalized Mathematics*, 10, 1998. http://mizar.org/JFM/Vol10/sin_cos.html.

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