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Properties of the Trigonometric Function

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Summary. This article introduces the monotone increasing and the monotone decreasing of *sinus* and *cosine*, and definitions of hyperbolic *sinus*, hyperbolic *cosine* and hyperbolic *tangent*, and some related formulas about them.

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The articles [9], [12], [1], [10], [2], [13], [6], [7], [11], [5], [8], [4], [14], and [3] provide the notation and terminology for this paper.

1. Monotone Increasing and Monotone Decreasing of Sinus and Cosine

We adopt the following convention: p, q, r, t_1 denote real numbers and n denotes a natural number.

The following propositions are true:

(1) If $p \geq 0$ and $r \geq 0$, then $p + r \geq 2 \cdot \sqrt{p \cdot r}$.

(2) sin is increasing on $]0, \frac{\pi}{2}[$.

(3) sin is decreasing on $]\frac{\pi}{2}, \pi[$.

(4) cos is decreasing on $]0, \frac{\pi}{2}[$.

(5) cos is decreasing on $]\frac{\pi}{2}, \pi[$.

(6) sin is decreasing on $]\pi, \frac{3}{2} \cdot \pi[$.

(7) sin is increasing on $]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$.

(8) cos is increasing on $]\pi, \frac{3}{2} \cdot \pi[$.

(9) cos is increasing on $]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$.

(10) $\sin(t_1) = \sin(2 \cdot \pi \cdot n + t_1)$.

(11) $\cos(t_1) = \cos(2 \cdot \pi \cdot n + t_1)$.

2. Hyperbolic Sinus, Hyperbolic Cosine and Hyperbolic Tangent

The partial function sinh from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

(Def. 1) $\operatorname{dom}\sinh = \mathbb{R}$ and for every real number d holds $\sinh(d) = \frac{\exp(d) - \exp(-d)}{2}$.

Let d be a number. The functor $\sinh d$ is defined as follows:

(Def. 2) $\sinh d = \sinh(d)$.

Let d be a number. One can check that $\sinh d$ is real.
Let d be a number. Then $\sinh d$ is a real number.
The partial function cosh from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 3) $\operatorname{dom}\cosh = \mathbb{R}$ and for every real number d holds $\cosh(d) = \frac{\exp(d) + \exp(-d)}{2}$.

Let d be a number. The functor $\cosh d$ is defined by:

(Def. 4) $\cosh d = \cosh(d)$.

Let d be a number. Note that $\cosh d$ is real.
Let d be a number. Then $\cosh d$ is a real number.
The partial function tanh from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

(Def. 5) $\operatorname{dom}\tanh = \mathbb{R}$ and for every real number d holds $\tanh(d) = \frac{\exp(d) - \exp(-d)}{\exp(d) + \exp(-d)}$.

Let d be a number. The functor $\tanh d$ is defined as follows:

(Def. 6) $\tanh d = \tanh(d)$.

Let d be a number. One can check that $\tanh d$ is real.
Let d be a number. Then $\tanh d$ is a real number.
We now state a number of propositions:

(12) $\exp(p+q) = \exp(p) \cdot \exp(q)$.

(13) $\exp(0) = 1$.

(14) $\cosh(p)^{2} - \sinh(p)^{2} = 1$ and $\cosh(p) \cdot \cosh(p) - \sinh(p) \cdot \sinh(p) = 1$.

(15) $\cosh(p) \neq 0$ and $\cosh(p) > 0$ and $\cosh(0) = 1$.

(16) $\sinh(0) = 0$.

(17) $\tanh(p) = \frac{\sinh(p)}{\cosh(p)}$.

(18) $\sinh(p)^{2} = \frac{1}{2} \cdot (\cosh(2 \cdot p) - 1)$ and $\cosh(p)^{2} = \frac{1}{2} \cdot (\cosh(2 \cdot p) + 1)$.

(19) $\cosh(-p) = \cosh(p)$ and $\sinh(-p) = -\sinh(p)$ and $\tanh(-p) = -\tanh(p)$.

(20) $\cosh(p+r) = \cosh(p) \cdot \cosh(r) + \sinh(p) \cdot \sinh(r)$ and $\cosh(p-r) = \cosh(p) \cdot \cosh(r) - \sinh(p) \cdot \sinh(r)$.

(21) $\sinh(p+r) = \sinh(p) \cdot \cosh(r) + \cosh(p) \cdot \sinh(r)$ and $\sinh(p-r) = \sinh(p) \cdot \cosh(r) - \cosh(p) \cdot \sinh(r)$.

(22) $\tanh(p+r) = \frac{\tanh(p) + \tanh(r)}{1 + \tanh(p) \cdot \tanh(r)}$ and $\tanh(p-r) = \frac{\tanh(p) - \tanh(r)}{1 - \tanh(p) \cdot \tanh(r)}$.

(23) $\sinh(2 \cdot p) = 2 \cdot \sinh(p) \cdot \cosh(p)$ and $\cosh(2 \cdot p) = 2 \cdot \cosh(p)^{2} - 1$ and $\tanh(2 \cdot p) = \frac{2 \cdot \tanh(p)}{1 + \tanh(p)^{2}}$.

(24) $\sinh(p)^{2} - \sinh(q)^{2} = \sinh(p+q) \cdot \sinh(p-q)$ and $\sinh(p+q) \cdot \sinh(p-q) = \cosh(p)^{2} - \cosh(q)^{2}$ and $\sinh(p)^{2} - \sinh(q)^{2} = \cosh(p)^{2} - \cosh(q)^{2}$.

(25) $\sinh(p)^2+\cosh(q)^2=\cosh(p+q)\cdot\cosh(p-q)$ and $\cosh(p+q)\cdot\cosh(p-q)=\cosh(p)^2+\sinh(q)^2$ and $\sinh(p)^2+\cosh(q)^2=\cosh(p)^2+\sinh(q)^2$.

(26) $\sinh(p)+\sinh(r)=2\cdot\sinh(\frac{p}{2}+\frac{r}{2})\cdot\cosh(\frac{p}{2}-\frac{r}{2})$ and $\sinh(p)-\sinh(r)=2\cdot\sinh(\frac{p}{2}-\frac{r}{2})\cdot\cosh(\frac{p}{2}+\frac{r}{2})$.

(27) $\cosh(p)+\cosh(r)=2\cdot\cosh(\frac{p}{2}+\frac{r}{2})\cdot\cosh(\frac{p}{2}-\frac{r}{2})$ and $\cosh(p)-\cosh(r)=2\cdot\sinh(\frac{p}{2}-\frac{r}{2})\cdot\sinh(\frac{p}{2}+\frac{r}{2})$.

(28) $\tanh(p)+\tanh(r)=\frac{\sinh(p+r)}{\cosh(p)\cdot\cosh(r)}$ and $\tanh(p)-\tanh(r)=\frac{\sinh(p-r)}{\cosh(p)\cdot\cosh(r)}$.

(29) $(\cosh(p)+\sinh(p))^n=\cosh(n\cdot p)+\sinh(n\cdot p)$.

One can verify the following observations:

* sinh is total,
* cosh is total, and
* tanh is total.

We now state a number of propositions:

(30) $\operatorname{dom}\sinh=\mathbb{R}$ and $\operatorname{dom}\cosh=\mathbb{R}$ and $\operatorname{dom}\tanh=\mathbb{R}$.

(31) sinh is differentiable in p and $\sinh'(p)=\cosh(p)$.

(32) cosh is differentiable in p and $\cosh'(p)=\sinh(p)$.

(33) tanh is differentiable in p and $\tanh'(p)=\frac{1}{\cosh(p)^2}$.

(34) sinh is differentiable on $\mathbb{R}$ and $\sinh'(p)=\cosh(p)$.

(35) cosh is differentiable on $\mathbb{R}$ and $\cosh'(p)=\sinh(p)$.

(36) tanh is differentiable on $\mathbb{R}$ and $\tanh'(p)=\frac{1}{\cosh(p)^2}$.

(37) $\cosh(p)\geq 1$.

(38) sinh is continuous in p.

(39) cosh is continuous in p.

(40) tanh is continuous in p.

(41) sinh is continuous on $\mathbb{R}$.

(42) cosh is continuous on $\mathbb{R}$.

(43) tanh is continuous on $\mathbb{R}$.

(44) $\tanh(p)<1$ and $\tanh(p)>-1$.

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