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# Trigonometric Functions and Existence of Circle Ratio

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**Summary.** In this article, we defined *sinus* and *cosine* as the real part and the imaginary part of the exponential function on complex, and also give their series expression. Then we proved the differentiablity of *sinus*, *cosine* and the exponential function of real. Finally, we showed the existence of the circle ratio, and some formulas of *sinus*, *cosine*.

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The articles [23], [26], [3], [24], [7], [8], [4], [19], [5], [11], [9], [20], [27], [18], [16], [2], [13], [6], [25], [17], [10], [21], [15], [12], [1], [14], and [22] provide the notation and terminology for this paper.

## 1. Some Definitions and Properties of Complex Sequence

For simplicity, we use the following convention: $p$, $q$, $t_1$, $t_2$, $t_3$ denote real numbers, $w$, $z$, $z_1$, $z_2$ denote elements of $\mathbb{C}$, $k$, $l$, $m$, $n$ denote natural numbers, $s_1$ denotes a complex sequence, and $r_1$ denotes a sequence of real numbers.

Let $m$, $k$ be natural numbers. The functor $\mathrm{CHK}(m,k)$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 2)[1] $\mathrm{CHK}(m,k) = \begin{cases} 1_{\mathbb{C}}, & \text{if } m \leq k, \\ 0_{\mathbb{C}}, & \text{otherwise.} \end{cases}$

The functor $\mathrm{RHK}(m,k)$ is defined as follows:

(Def. 3) $\mathrm{RHK}(m,k) = \begin{cases} 1, & \text{if } m \leq k, \\ 0, & \text{otherwise.} \end{cases}$

Let $m$, $k$ be natural numbers. One can check that $\mathrm{RHK}(m,k)$ is real.

Let $m$, $k$ be natural numbers. Then $\mathrm{RHK}(m,k)$ is a real number.

In this article we present several logical schemes. The scheme *ExComplex CASE* deals with a binary functor $\mathcal{F}$ yielding an element of $\mathbb{C}$, and states that:

For every $k$ there exists $s_1$ such that for every $n$ holds if $n \leq k$, then $s_1(n) = \mathcal{F}(k,n)$ and if $n > k$, then $s_1(n) = 0_{\mathbb{C}}$

for all values of the parameter.

The scheme *ExReal CASE* deals with a binary functor $\mathcal{F}$ yielding a real number, and states that:

For every $k$ there exists $r_1$ such that for every $n$ holds if $n \leq k$, then $r_1(n) = \mathcal{F}(k,n)$ and if $n > k$, then $r_1(n) = 0$

[1] The definition (Def. 1) has been removed.

for all values of the parameter.

The complex sequence Prod_complex_n is defined by:

(Def. 4) $(\text{Prod\_complex\_n})(0) = 1_{\mathbb{C}}$ and for every $n$ holds $(\text{Prod\_complex\_n})(n+1) = (\text{Prod\_complex\_n})(n) \cdot ((n+1)+0i)$.

The sequence Prod_real_n of real numbers is defined as follows:

(Def. 5) $(\text{Prod\_real\_n})(0) = 1$ and for every $n$ holds $(\text{Prod\_real\_n})(n+1) = (\text{Prod\_real\_n})(n) \cdot (n+1)$.

Let $n$ be a natural number. The functor $n!_{\mathbb{C}}$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 6) $n!_{\mathbb{C}} = (\text{Prod\_complex\_n})(n)$.

Let $n$ be a natural number. Then $n!$ is a real number and it can be characterized by the condition:

(Def. 7) $n! = (\text{Prod\_real\_n})(n)$.

Let $z$ be an element of $\mathbb{C}$. The functor $z\,\text{ExpSeq}$ yielding a complex sequence is defined as follows:

(Def. 8) For every $n$ holds $z\,\text{ExpSeq}(n) = \frac{z^n_{\mathbb{N}}}{n!_{\mathbb{C}}}$.

Let $a$ be a real number. The functor $a\,\text{ExpSeq}$ yields a sequence of real numbers and is defined by:

(Def. 9) For every $n$ holds $a\,\text{ExpSeq}(n) = \frac{a^n}{n!}$.

One can prove the following three propositions:

(1) If $0 < n$, then $n+0i \neq 0_{\mathbb{C}}$ and $0!_{\mathbb{C}} = 1_{\mathbb{C}}$ and $n!_{\mathbb{C}} \neq 0_{\mathbb{C}}$ and $(n+1)!_{\mathbb{C}} = n!_{\mathbb{C}} \cdot ((n+1)+0i)$.

(2) $n! \neq 0$ and $(n+1)! = n! \cdot (n+1)$.

(3) For every $k$ such that $0 < k$ holds $(k -' 1)!_{\mathbb{C}} \cdot (k+0i) = k!_{\mathbb{C}}$ and for all $m$, $k$ such that $k \leq m$ holds $(m -' k)!_{\mathbb{C}} \cdot (((m+1)-k)+0i) = ((m+1) -' k)!_{\mathbb{C}}$.

Let $n$ be a natural number. The functor $\text{Coef}\,n$ yields a complex sequence and is defined as follows:

(Def. 10) For every natural number $k$ holds if $k \leq n$, then $(\text{Coef}\,n)(k) = \frac{n!_{\mathbb{C}}}{k!_{\mathbb{C}} \cdot (n -' k)!_{\mathbb{C}}}$ and if $k > n$, then $(\text{Coef}\,n)(k) = 0_{\mathbb{C}}$.

Let $n$ be a natural number. The functor $\text{Coef\_e}\,n$ yielding a complex sequence is defined as follows:

(Def. 11) For every natural number $k$ holds if $k \leq n$, then $(\text{Coef\_e}\,n)(k) = \frac{1_{\mathbb{C}}}{k!_{\mathbb{C}} \cdot (n -' k)!_{\mathbb{C}}}$ and if $k > n$, then $(\text{Coef\_e}\,n)(k) = 0_{\mathbb{C}}$.

Let us consider $s_1$. The functor $\text{Sift}\,s_1$ yielding a complex sequence is defined by:

(Def. 12) $(\text{Sift}\,s_1)(0) = 0_{\mathbb{C}}$ and for every natural number $k$ holds $(\text{Sift}\,s_1)(k+1) = s_1(k)$.

Let us consider $n$ and let $z$, $w$ be elements of $\mathbb{C}$. The functor $\text{Expan}(n,z,w)$ yields a complex sequence and is defined by:

(Def. 13) For every natural number $k$ holds if $k \leq n$, then $(\text{Expan}(n,z,w))(k) = (\text{Coef}\,n)(k) \cdot z^k_{\mathbb{N}} \cdot w^{n -' k}_{\mathbb{N}}$ and if $n < k$, then $(\text{Expan}(n,z,w))(k) = 0_{\mathbb{C}}$.

Let us consider $n$ and let $z$, $w$ be elements of $\mathbb{C}$. The functor $\text{Expan\_e}(n,z,w)$ yields a complex sequence and is defined as follows:

(Def. 14) For every natural number $k$ holds if $k \leq n$, then $(\text{Expan\_e}(n,z,w))(k) = (\text{Coef\_e}\,n)(k) \cdot z^k_{\mathbb{N}} \cdot w^{n -' k}_{\mathbb{N}}$ and if $n < k$, then $(\text{Expan\_e}(n,z,w))(k) = 0_{\mathbb{C}}$.

Let us consider $n$ and let $z$, $w$ be elements of $\mathbb{C}$. The functor $\mathrm{Alfa}(n,z,w)$ yields a complex sequence and is defined as follows:

(Def. 15) For every natural number $k$ holds if $k \leq n$, then $(\mathrm{Alfa}(n,z,w))(k) = z\,\mathrm{ExpSeq}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n -' k)$ and if $n < k$, then $(\mathrm{Alfa}(n,z,w))(k) = 0_{\mathbb{C}}$.

Let $a$, $b$ be real numbers and let $n$ be a natural number. The functor $\mathrm{Conj}(n,a,b)$ yielding a sequence of real numbers is defined as follows:

(Def. 16) For every natural number $k$ holds if $k \leq n$, then $(\mathrm{Conj}(n,a,b))(k) = a\,\mathrm{ExpSeq}(k) \cdot ((\sum_{\alpha=0}^{\kappa} b\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} b\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n -' k))$ and if $n < k$, then $(\mathrm{Conj}(n,a,b))(k) = 0$.

Let $z$, $w$ be elements of $\mathbb{C}$ and let $n$ be a natural number. The functor $\mathrm{Conj}(n,z,w)$ yields a complex sequence and is defined as follows:

(Def. 17) For every natural number $k$ holds if $k \leq n$, then $(\mathrm{Conj}(n,z,w))(k) = z\,\mathrm{ExpSeq}(k) \cdot ((\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n -' k))$ and if $n < k$, then $(\mathrm{Conj}(n,z,w))(k) = 0_{\mathbb{C}}$.

The following propositions are true:

(4) $z\,\mathrm{ExpSeq}(n+1) = \frac{z\,\mathrm{ExpSeq}(n)\cdot z}{(n+1)+0i}$ and $z\,\mathrm{ExpSeq}(0) = 1_{\mathbb{C}}$ and $|z\,\mathrm{ExpSeq}(n)| = |z|\,\mathrm{ExpSeq}(n)$.

(5) If $0 < k$, then $(\mathrm{Sift}\,s_1)(k) = s_1(k -' 1)$.

(6) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Sift}\,s_1)(\alpha))_{\kappa\in\mathbb{N}}(k) + s_1(k)$.

(7) $(z+w)_{\mathbb{N}}^{n} = (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan}(n,z,w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(8) $\mathrm{Expan\_e}(n,z,w) = \frac{1_{\mathbb{C}}}{n!_{\mathbb{C}}}\ \mathrm{Expan}(n,z,w)$.

(9) $\frac{(z+w)_{\mathbb{N}}^{n}}{n!_{\mathbb{C}}} = (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan\_e}(n,z,w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(10) $0_{\mathbb{C}}\,\mathrm{ExpSeq}$ is absolutely summable and $\sum(0_{\mathbb{C}}\,\mathrm{ExpSeq}) = 1_{\mathbb{C}}$.

Let us consider $z$. Note that $z\,\mathrm{ExpSeq}$ is absolutely summable.

One can prove the following propositions:

(11) $z\,\mathrm{ExpSeq}(0) = 1_{\mathbb{C}}$ and $(\mathrm{Expan}(0,z,w))(0) = 1_{\mathbb{C}}$.

(12) If $l \leq k$, then $(\mathrm{Alfa}(k+1,z,w))(l) = (\mathrm{Alfa}(k,z,w))(l) + (\mathrm{Expan\_e}(k+1,z,w))(l)$.

(13) $(\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(k+1,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(k,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k) + (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan\_e}(k+1,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(14) $z\,\mathrm{ExpSeq}(k) = (\mathrm{Expan\_e}(k,z,w))(k)$.

(15) $(\sum_{\alpha=0}^{\kappa} z+w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(n,z,w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(16) $(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) - (\sum_{\alpha=0}^{\kappa} z+w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Conj}(k,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(17) $|(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)| \leq (\sum_{\alpha=0}^{\kappa}|z|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)$ and $(\sum_{\alpha=0}^{\kappa}|z|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) \leq \sum(|z|\,\mathrm{ExpSeq})$ and $|(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)| \leq \sum(|z|\,\mathrm{ExpSeq})$.

(18) $1 \leq \sum(|z|\,\mathrm{ExpSeq})$.

(19) $0 \leq |z|\,\mathrm{ExpSeq}(n)$.

(20) $|(\sum_{\alpha=0}^{\kappa}|z|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}|z|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)$ and if $n \leq m$, then $|(\sum_{\alpha=0}^{\kappa}|z|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}|z|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}|z|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}|z|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(21) $|(\sum_{\alpha=0}^{\kappa} |\mathrm{Conj}(k,z,w)|(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa} |\mathrm{Conj}(k,z,w)|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(22) For every real number $p$ such that $p > 0$ there exists $n$ such that for every $k$ such that $n \leq k$ holds $|(\sum_{\alpha=0}^{\kappa} |\mathrm{Conj}(k,z,w)|(\alpha))_{\kappa\in\mathbb{N}}(k)| < p$.

(23) For every $s_1$ such that for every $k$ holds $s_1(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Conj}(k,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k)$ holds $s_1$ is convergent and $\lim s_1 = 0_{\mathbb{C}}$.

## 2. Definition of Exponential Function on Complex

The partial function exp from $\mathbb{C}$ to $\mathbb{C}$ is defined as follows:

(Def. 18) $\mathrm{dom}\exp = \mathbb{C}$ and for every element $z$ of $\mathbb{C}$ holds $\exp(z) = \sum(z\,\mathrm{ExpSeq})$.

Let us consider $z$. The functor $\exp z$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 19) $\exp z = \exp(z)$.

The following proposition is true

(24) For all $z_1$, $z_2$ holds $\exp(z_1 + z_2) = \exp z_1 \cdot \exp z_2$.

## 3. Definition of Sinus, Cosine, and Exponential Function on $\mathbb{R}$

The partial function sin from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 20) $\mathrm{dom}\sin = \mathbb{R}$ and for every element $d$ of $\mathbb{R}$ holds $\sin(d) = \Im(\sum(0 + d i\,\mathrm{ExpSeq}))$.

Let $t_1$ be a real number. The functor $\sin t_1$ is defined as follows:

(Def. 21) $\sin t_1 = \sin(t_1)$.

Let $t_1$ be a real number. One can check that $\sin t_1$ is real.
Let $t_1$ be a real number. Then $\sin t_1$ is a real number.
Next we state the proposition

(25) sin is a function from $\mathbb{R}$ into $\mathbb{R}$.

In the sequel $d$ denotes a real number.
The partial function cos from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 22) $\mathrm{dom}\cos = \mathbb{R}$ and for every $d$ holds $\cos(d) = \Re(\sum(0 + d i\,\mathrm{ExpSeq}))$.

Let $t_1$ be a real number. The functor $\cos t_1$ is defined by:

(Def. 23) $\cos t_1 = \cos(t_1)$.

Let $t_1$ be a real number. Note that $\cos t_1$ is real.
Let $t_1$ be a real number. Then $\cos t_1$ is a real number.
Next we state several propositions:

(26) cos is a function from $\mathbb{R}$ into $\mathbb{R}$.

(27) $\mathrm{dom}\sin = \mathbb{R}$ and $\mathrm{dom}\cos = \mathbb{R}$.

(28) $\exp(0 + t_1 i) = \cos t_1 + \sin t_1 i$.

(29) $\overline{\exp(0 + t_1 i)} = \exp(-(0 + t_1 i))$.

(30) $|\exp(0 + t_1 i)| = 1$ and $|\sin t_1| \leq 1$ and $|\cos t_1| \leq 1$.

(31) $\cos(t_1)^2 + \sin(t_1)^2 = 1$ and $\cos(t_1) \cdot \cos(t_1) + \sin(t_1) \cdot \sin(t_1) = 1$.

(32) $(\cos t_1)^2 + (\sin t_1)^2 = 1$ and $\cos t_1 \cdot \cos t_1 + \sin t_1 \cdot \sin t_1 = 1$.

(33) $\cos(0) = 1$ and $\sin(0) = 0$ and $\cos(-t_1) = \cos(t_1)$ and $\sin(-t_1) = -\sin(t_1)$.

(34) $\cos 0 = 1$ and $\sin 0 = 0$ and $\cos(-t_1) = \cos t_1$ and $\sin(-t_1) = -\sin t_1$.

Let $t_1$ be a real number. The functor $t_1$ P_sin yielding a sequence of real numbers is defined by:

(Def. 24) For every $n$ holds $t_1\,\mathrm{P\_sin}(n) = \frac{(-1)^n \cdot {t_1}^{2 \cdot n+1}}{(2 \cdot n+1)!}$.

The functor $t_1$ P_cos yields a sequence of real numbers and is defined by:

(Def. 25) For every $n$ holds $t_1\,\mathrm{P\_cos}(n) = \frac{(-1)^n \cdot {t_1}^{2 \cdot n}}{(2 \cdot n)!}$.

Next we state a number of propositions:

(35) For all $z$, $k$ holds $z^{2 \cdot k}_{\mathbb{N}} = (z^k_{\mathbb{N}})^2_{\mathbb{N}}$ and $z^{2 \cdot k}_{\mathbb{N}} = (z^2_{\mathbb{N}})^k_{\mathbb{N}}$.

(36) For all $k$, $t_1$ holds $(0 + t_1 i)^{2 \cdot k}_{\mathbb{N}} = (-1)^k \cdot {t_1}^{2 \cdot k} + 0i$ and $(0 + t_1 i)^{2 \cdot k+1}_{\mathbb{N}} = 0 + ((-1)^k \cdot {t_1}^{2 \cdot k+1})i$.

(37) For every $n$ holds $n!_{\mathbb{C}} = n! + 0i$.

(38) For all $t_1$, $n$ holds $(\sum_{\alpha=0}^{\kappa} t_1\,\mathrm{P\_sin}(\alpha))_{\kappa \in \mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa} \Im(0 + t_1 i\,\mathrm{ExpSeq})(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot n + 1)$ and $(\sum_{\alpha=0}^{\kappa} t_1\,\mathrm{P\_cos}(\alpha))_{\kappa \in \mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa} \Re(0 + t_1 i\,\mathrm{ExpSeq})(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot n)$.

(39) For every $t_1$ holds $(\sum_{\alpha=0}^{\kappa} t_1\,\mathrm{P\_sin}(\alpha))_{\kappa \in \mathbb{N}}$ is convergent and $\sum(t_1\,\mathrm{P\_sin}) = \Im(\sum(0 + t_1 i\,\mathrm{ExpSeq}))$ and $(\sum_{\alpha=0}^{\kappa} t_1\,\mathrm{P\_cos}(\alpha))_{\kappa \in \mathbb{N}}$ is convergent and $\sum(t_1\,\mathrm{P\_cos}) = \Re(\sum(0 + t_1 i\,\mathrm{ExpSeq}))$.

(40) For every $t_1$ holds $\cos(t_1) = \sum(t_1\,\mathrm{P\_cos})$ and $\sin(t_1) = \sum(t_1\,\mathrm{P\_sin})$.

(41) For all $p$, $t_1$, $r_1$ such that $r_1$ is convergent and $\lim r_1 = t_1$ and for every $n$ holds $r_1(n) \geq p$ holds $t_1 \geq p$.

(42) For all $n$, $k$, $m$ such that $n < k$ holds $m! > 0$ and $n! \leq k!$.

(43) For all $t_1$, $n$, $k$ such that $0 \leq t_1$ and $t_1 \leq 1$ and $n \leq k$ holds ${t_1}^k \leq {t_1}^n$.

(44) For all $t_1$, $n$ holds $(t_1 + 0i)^n_{\mathbb{N}} = {t_1}^n + 0i$.

(45) For all $t_1$, $n$ holds $\frac{(t_1+0i)^n_{\mathbb{N}}}{n!_{\mathbb{C}}} = \frac{{t_1}^n}{n!} + 0i$.

(46) $\Im(\sum(p + 0i\,\mathrm{ExpSeq})) = 0$.

(47) $\cos(1) > 0$ and $\sin(1) > 0$ and $\cos(1) < \sin(1)$.

(48) For every $t_1$ holds $t_1\,\mathrm{ExpSeq} = \Re(t_1 + 0i\,\mathrm{ExpSeq})$.

(49) For every $t_1$ holds $t_1\,\mathrm{ExpSeq}$ is summable and $\sum(t_1\,\mathrm{ExpSeq}) = \Re(\sum(t_1 + 0i\,\mathrm{ExpSeq}))$.

(50) For all $p$, $q$ holds $\sum(p + q\,\mathrm{ExpSeq}) = \sum(p\,\mathrm{ExpSeq}) \cdot \sum(q\,\mathrm{ExpSeq})$.

The partial function exp from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 26) $\mathrm{dom}\exp = \mathbb{R}$ and for every real number $d$ holds $\exp(d) = \sum(d\,\mathrm{ExpSeq})$.

Let $t_1$ be a real number. The functor $\exp t_1$ is defined as follows:

(Def. 27) $\exp t_1 = \exp(t_1)$.

Let $t_1$ be a real number. One can check that $\exp t_1$ is real.
Let $t_1$ be a real number. Then $\exp t_1$ is a real number.
The following propositions are true:

(51) $\mathrm{dom}\exp = \mathbb{R}$.

(53)[2] For every $t_1$ holds $\exp(t_1) = \Re(\sum(t_1 + 0i\,\mathrm{ExpSeq}))$.

(54) $\exp(t_1 + 0i) = \exp t_1 + 0i$.

(55) $\exp(p+q) = \exp p \cdot \exp q$.

(56) $\exp 0 = 1$.

(57) For every $t_1$ such that $t_1 > 0$ holds $\exp(t_1) \geq 1$.

(58) For every $t_1$ such that $t_1 < 0$ holds $0 < \exp(t_1)$ and $\exp(t_1) \leq 1$.

(59) For every $t_1$ holds $\exp(t_1) > 0$.

(60) For every $t_1$ holds $\exp t_1 > 0$.

## 4. Differential of Sinus, Cosine, and Exponential Function

Let $z$ be an element of $\mathbb{C}$. The functor $z\,\mathrm{P\_dt}$ yielding a complex sequence is defined as follows:

(Def. 28) For every $n$ holds $z\,\mathrm{P\_dt}(n) = \frac{z_{\mathbb{N}}^{n+1}}{(n+2)!_{\mathbb{C}}}$.

The functor $z\,\mathrm{P\_t}$ yielding a complex sequence is defined as follows:

(Def. 29) For every $n$ holds $z\,\mathrm{P\_t}(n) = \frac{z_{\mathbb{N}}^{n}}{(n+2)!_{\mathbb{C}}}$.

Next we state a number of propositions:

(61) For every $z$ holds $z\,\mathrm{P\_dt}$ is absolutely summable.

(62) For every $z$ holds $z \cdot \sum(z\,\mathrm{P\_dt}) = \sum(z\,\mathrm{ExpSeq}) - 1_{\mathbb{C}} - z$.

(63) For every $p$ such that $p > 0$ there exists $q$ such that $q > 0$ and for every $z$ such that $|z| < q$ holds $|\sum(z\,\mathrm{P\_dt})| < p$.

(64) For all $z$, $z_1$ holds $\sum(z_1 + z\,\mathrm{ExpSeq}) - \sum(z_1\,\mathrm{ExpSeq}) = \sum(z_1\,\mathrm{ExpSeq}) \cdot z + z \cdot \sum(z\,\mathrm{P\_dt}) \cdot \sum(z_1\,\mathrm{ExpSeq})$.

(65) For all $p$, $q$ holds $\cos(p+q) - \cos(p) = -q \cdot \sin(p) - q \cdot \Im(\sum(0 + qi\,\mathrm{P\_dt}) \cdot (\cos(p) + \sin(p)i))$.

(66) For all $p$, $q$ holds $\sin(p+q) - \sin(p) = q \cdot \cos(p) + q \cdot \Re(\sum(0 + qi\,\mathrm{P\_dt}) \cdot (\cos(p) + \sin(p)i))$.

(67) For all $p$, $q$ holds $\exp(p+q) - \exp(p) = q \cdot \exp(p) + q \cdot \exp(p) \cdot \Re(\sum(q + 0i\,\mathrm{P\_dt}))$.

(68) For every $p$ holds cos is differentiable in $p$ and $\cos'(p) = -\sin(p)$.

(69) For every $p$ holds sin is differentiable in $p$ and $\sin'(p) = \cos(p)$.

(70) For every $p$ holds exp is differentiable in $p$ and $\exp'(p) = \exp(p)$.

(71) exp is differentiable on $\mathbb{R}$ and for every $t_1$ such that $t_1 \in \mathbb{R}$ holds $\exp'(t_1) = \exp(t_1)$.

(72) cos is differentiable on $\mathbb{R}$ and for every $t_1$ such that $t_1 \in \mathbb{R}$ holds $\cos'(t_1) = -\sin(t_1)$.

(73) sin is differentiable on $\mathbb{R}$ and for every $t_1$ holds $\sin'(t_1) = \cos(t_1)$.

(74) For every $t_1$ such that $t_1 \in [0,1]$ holds $0 < \cos(t_1)$ and $\cos(t_1) \geq \frac{1}{2}$.

(75) $[0,1] \subseteq \mathrm{dom}(\frac{\sin}{\cos})$ and $]0,1[ \subseteq \mathrm{dom}(\frac{\sin}{\cos})$.

(76) $\frac{\sin}{\cos}$ is continuous on $[0,1]$.

(77) For all $t_2$, $t_3$ such that $t_2 \in ]0,1[$ and $t_3 \in ]0,1[$ and $(\frac{\sin}{\cos})(t_2) = (\frac{\sin}{\cos})(t_3)$ holds $t_2 = t_3$.

[2] The proposition (52) has been removed.

## 5. Existence of Circle Ratio

The real number $\pi$ is defined as follows:

(Def. 30) $(\frac{\sin}{\cos})(\frac{\pi}{4}) = 1$ and $\pi \in ]0,4[$.

$\pi$ is a real number.
The following proposition is true

(78) $\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4})$.

## 6. Formulas of Sinus, Cosine

One can prove the following propositions:

(79) $\sin(t_2+t_3) = \sin(t_2)\cdot\cos(t_3)+\cos(t_2)\cdot\sin(t_3)$ and $\cos(t_2+t_3) = \cos(t_2)\cdot\cos(t_3)-\sin(t_2)\cdot\sin(t_3)$.

(80) $\sin(t_2+t_3) = \sin t_2\cdot\cos t_3+\cos t_2\cdot\sin t_3$ and $\cos(t_2+t_3) = \cos t_2\cdot\cos t_3-\sin t_2\cdot\sin t_3$.

(81) $\cos(\frac{\pi}{2}) = 0$ and $\sin(\frac{\pi}{2}) = 1$ and $\cos(\pi) = -1$ and $\sin(\pi) = 0$ and $\cos(\pi+\frac{\pi}{2}) = 0$ and $\sin(\pi+\frac{\pi}{2}) = -1$ and $\cos(2\cdot\pi) = 1$ and $\sin(2\cdot\pi) = 0$.

(82) $\cos(\frac{\pi}{2}) = 0$ and $\sin(\frac{\pi}{2}) = 1$ and $\cos\pi = -1$ and $\sin\pi = 0$ and $\cos(\pi+\frac{\pi}{2}) = 0$ and $\sin(\pi+\frac{\pi}{2}) = -1$ and $\cos(2\cdot\pi) = 1$ and $\sin(2\cdot\pi) = 0$.

(83) $\sin(t_1+2\cdot\pi) = \sin(t_1)$ and $\cos(t_1+2\cdot\pi) = \cos(t_1)$ and $\sin(\frac{\pi}{2}-t_1) = \cos(t_1)$ and $\cos(\frac{\pi}{2}-t_1) = \sin(t_1)$ and $\sin(\frac{\pi}{2}+t_1) = \cos(t_1)$ and $\cos(\frac{\pi}{2}+t_1) = -\sin(t_1)$ and $\sin(\pi+t_1) = -\sin(t_1)$ and $\cos(\pi+t_1) = -\cos(t_1)$.

(84) $\sin(t_1+2\cdot\pi) = \sin t_1$ and $\cos(t_1+2\cdot\pi) = \cos t_1$ and $\sin(\frac{\pi}{2}-t_1) = \cos t_1$ and $\cos(\frac{\pi}{2}-t_1) = \sin t_1$ and $\sin(\frac{\pi}{2}+t_1) = \cos t_1$ and $\cos(\frac{\pi}{2}+t_1) = -\sin t_1$ and $\sin(\pi+t_1) = -\sin t_1$ and $\cos(\pi+t_1) = -\cos t_1$.

(85) For every $t_1$ such that $t_1 \in ]0,\frac{\pi}{2}[$ holds $\cos(t_1) > 0$.

(86) For every $t_1$ such that $t_1 \in ]0,\frac{\pi}{2}[$ holds $\cos t_1 > 0$.

## References


[1] Agnieszka Banachowicz and Anna Winnicka. Complex sequences. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/comseq_1.html.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[3] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[6] Czesław Byliński. The complex numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/complex1.html.

[7] Library Committee. Introduction to arithmetic. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/arytm_0.html.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[9] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_2.html.

[10] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seqm_3.html.

[11] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_1.html`.

[12] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rfunct_1.html`.

[13] Rafał Kwiatek. Factorial and Newton coefficients. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/newton.html`.

[14] Adam Naumowicz. Conjugate sequences, bounded complex sequences and convergent complex sequences. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/comseq_2.html`.

[15] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/binarith.html`.

[16] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/absvalue.html`.

[17] Konrad Raczkowski. Integer and rational exponents. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/prepower.html`.

[18] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/series_1.html`.

[19] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/fcont_1.html`.

[20] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/fdiff_1.html`.

[21] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rcomp_1.html`.

[22] Yasunari Shidama and Artur Korniłowicz. Convergence and the limit of complex sequences. Series. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/comseq_3.html`.

[23] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[24] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[25] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/square_1.html`.

[26] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[27] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.



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