JOURNAL OF FORMALIZED MATHEMATICS
Volume 4, Released 1992, Published 2003
Inst. of Computer Science, Univ. of Białystok

Functional Sequence from a Domain to a Domain

Beata Perkowska
Warsaw University
Białystok

Summary. Definitions of functional sequences and basic operations on functional sequences from a domain to a domain, point and uniform convergence, limit of functional sequence from a domain to the set of real numbers and facts about properties of the limit of functional sequences are proved.

MML Identifier: SEQFUNC.

WWW: http://mizar.org/JFM/Vol4/seqfunc.html

The articles [10], [12], [1], [11], [4], [13], [2], [14], [3], [7], [8], [6], [5], and [9] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: D, D_1, D_2 denote non empty sets, n, k denote natural numbers, p, r denote real numbers, and f denotes a function.

Let us consider D_1, D_2. A function is called a sequence of partial functions from D_1 into D_2 if:

(Def. 1) $\operatorname{dom} \mathrm{it} = \mathbb{N}$ and $\operatorname{rng} \mathrm{it} \subseteq D_1 \dot{\rightarrow} D_2$.

In the sequel F, F_1, F_2 are sequences of partial functions from D_1 into D_2.

Let us consider D_1, D_2, F, n. Then $F(n)$ is a partial function from D_1 to D_2.

In the sequel G, H, H_1, H_2, J denote sequences of partial functions from D into $\mathbb{R}$.

Next we state two propositions:

(1) f is a sequence of partial functions from D_1 into D_2 if and only if $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n)$ is a partial function from D_1 to D_2.

(2) For all F_1, F_2 such that for every n holds $F_1(n) = F_2(n)$ holds $F_1 = F_2$.

The scheme *ExFuncSeq* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a partial function from $\mathcal{A}$ to $\mathcal{B}$, and states that:

There exists a sequence G of partial functions from $\mathcal{A}$ into $\mathcal{B}$ such that for every n holds $G(n) = \mathcal{F}(n)$

for all values of the parameters.

Let us consider D, H, r. The functor $r\,H$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined by:

(Def. 2) For every n holds $(r\,H)(n) = r\,H(n)$.

Let us consider D, H. The functor H^{-1} yielding a sequence of partial functions from D into $\mathbb{R}$ is defined as follows:

(Def. 3) For every n holds $H^{-1}(n) = \frac{1}{H(n)}$.

The functor $-H$ yielding a sequence of partial functions from D into $\mathbb{R}$ is defined as follows:

(Def. 4) For every n holds $(-H)(n) = -H(n)$.

The functor $|H|$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined as follows:

(Def. 5) For every n holds $|H|(n) = |H(n)|$.

Let us consider D, G, H. The functor $G+H$ yielding a sequence of partial functions from D into $\mathbb{R}$ is defined as follows:

(Def. 6) For every n holds $(G+H)(n) = G(n)+H(n)$.

Let us consider D, G, H. The functor $G-H$ yielding a sequence of partial functions from D into $\mathbb{R}$ is defined by:

(Def. 7) $G-H = G+-H$.

Let us consider D, G, H. The functor $G\,H$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined by:

(Def. 8) For every n holds $(G\,H)(n) = G(n)\,H(n)$.

Let us consider D, H, G. The functor $\frac{G}{H}$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined by:

(Def. 9) $\frac{G}{H} = G\,H^{-1}$.

The following propositions are true:

(3) $H_1 = \frac{G}{H}$ iff for every n holds $H_1(n) = \frac{G(n)}{H(n)}$.

(4) $H_1 = G-H$ iff for every n holds $H_1(n) = G(n)-H(n)$.

(5) $G+H = H+G$ and $(G+H)+J = G+(H+J)$.

(6) $G\,H = H\,G$ and $(G\,H)\,J = G\,(H\,J)$.

(7) $(G+H)\,J = G\,J+H\,J$ and $J\,(G+H) = J\,G+J\,H$.

(8) $-H = (-1)\,H$.

(9) $(G-H)\,J = G\,J-H\,J$ and $J\,G-J\,H = J\,(G-H)$.

(10) $r\,(G+H) = r\,G+r\,H$ and $r\,(G-H) = r\,G-r\,H$.

(11) $(r \cdot p)\,H = r\,(p\,H)$.

(12) $1\,H = H$.

(13) $--H = H$.

(14) $G^{-1}\,H^{-1} = (G\,H)^{-1}$.

(15) If $r \neq 0$, then $(r\,H)^{-1} = r^{-1}\,H^{-1}$.

(16) $|H|^{-1} = |H^{-1}|$.

(17) $|G\,H| = |G|\,|H|$.

(18) $|\frac{G}{H}| = \frac{|G|}{|H|}$.

(19) $|r\,H| = |r|\,|H|$.

In the sequel x denotes an element of D, X, Y denote sets, and f denotes a partial function from D to $\mathbb{R}$.

Let us consider D_1, D_2, F, X. We say that X is common for elements of F if and only if:

(Def. 10) $X \neq \emptyset$ and for every n holds $X \subseteq \mathrm{dom}\, F(n)$.

Let us consider D, H, x. The functor $H\#x$ yields a sequence of real numbers and is defined by:

(Def. 11) For every n holds $(H\#x)(n) = H(n)(x)$.

Let us consider D, H, X. We say that H is point-convergent on X if and only if the conditions (Def. 12) are satisfied.

(Def. 12)(i) X is common for elements of H, and

(ii) there exists f such that $X = \mathrm{dom}\, f$ and for every x such that $x \in X$ and for every p such that $p > 0$ there exists k such that for every n such that $n \geq k$ holds $|H(n)(x) - f(x)| < p$.

We now state two propositions:

(20) H is point-convergent on X if and only if the following conditions are satisfied:

(i) X is common for elements of H, and

(ii) there exists f such that $X = \mathrm{dom}\, f$ and for every x such that $x \in X$ holds $H\#x$ is convergent and $\lim(H\#x) = f(x)$.

(21) H is point-convergent on X if and only if the following conditions are satisfied:

(i) X is common for elements of H, and

(ii) for every x such that $x \in X$ holds $H\#x$ is convergent.

Let us consider D, H, X. We say that H is uniform-convergent on X if and only if the conditions (Def. 13) are satisfied.

(Def. 13)(i) X is common for elements of H, and

(ii) there exists f such that $X = \mathrm{dom}\, f$ and for every p such that $p > 0$ there exists k such that for all n, x such that $n \geq k$ and $x \in X$ holds $|H(n)(x) - f(x)| < p$.

Let us consider D, H, X. Let us assume that H is point-convergent on X. The functor $\lim_X H$ yields a partial function from D to $\mathbb{R}$ and is defined as follows:

(Def. 14) $\mathrm{dom} \lim_X H = X$ and for every x such that $x \in \mathrm{dom} \lim_X H$ holds $(\lim_X H)(x) = \lim(H\#x)$.

We now state a number of propositions:

(22) Suppose H is point-convergent on X. Then $f = \lim_X H$ if and only if the following conditions are satisfied:

(i) $\mathrm{dom}\, f = X$, and

(ii) for every x such that $x \in X$ and for every p such that $p > 0$ there exists k such that for every n such that $n \geq k$ holds $|H(n)(x) - f(x)| < p$.

(23) If H is uniform-convergent on X, then H is point-convergent on X.

(24) If $Y \subseteq X$ and $Y \neq \emptyset$ and X is common for elements of H, then Y is common for elements of H.

(25) If $Y \subseteq X$ and $Y \neq \emptyset$ and H is point-convergent on X, then H is point-convergent on Y and $\lim_X H \restriction Y = \lim_Y H$.

(26) If $Y \subseteq X$ and $Y \neq \emptyset$ and H is uniform-convergent on X, then H is uniform-convergent on Y.

(27) If X is common for elements of H, then for every x such that $x \in X$ holds $\{x\}$ is common for elements of H.

(28) If H is point-convergent on X, then for every x such that $x \in X$ holds $\{x\}$ is common for elements of H.

(29) Suppose $\{x\}$ is common for elements of H_1 and $\{x\}$ is common for elements of H_2. Then $H_1\#x + H_2\#x = (H_1 + H_2)\#x$ and $H_1\#x - H_2\#x = (H_1 - H_2)\#x$ and $(H_1\#x)\,(H_2\#x) = (H_1\,H_2)\#x$.

(30) If $\{x\}$ is common for elements of H, then $|H|\#x = |H\#x|$ and $(-H)\#x = -H\#x$.

(31) If $\{x\}$ is common for elements of H, then $(r\,H)\#x = r\,(H\#x)$.

(32) Suppose X is common for elements of H_1 and common for elements of H_2. Let given x. If $x \in X$, then $H_1\#x + H_2\#x = (H_1 + H_2)\#x$ and $H_1\#x - H_2\#x = (H_1 - H_2)\#x$ and $(H_1\#x)\,(H_2\#x) = (H_1\,H_2)\#x$.

(33) If X is common for elements of H, then for every x such that $x \in X$ holds $|H|\#x = |H\#x|$ and $(-H)\#x = -H\#x$.

(34) If X is common for elements of H, then for every x such that $x \in X$ holds $(r\,H)\#x = r\,(H\#x)$.

(35) Suppose H_1 is point-convergent on X and H_2 is point-convergent on X. Let given x. If $x \in X$, then $H_1\#x + H_2\#x = (H_1 + H_2)\#x$ and $H_1\#x - H_2\#x = (H_1 - H_2)\#x$ and $(H_1\#x)\,(H_2\#x) = (H_1\,H_2)\#x$.

(36) If H is point-convergent on X, then for every x such that $x \in X$ holds $|H|\#x = |H\#x|$ and $(-H)\#x = -H\#x$.

(37) If H is point-convergent on X, then for every x such that $x \in X$ holds $(r\,H)\#x = r\,(H\#x)$.

(38) Suppose X is common for elements of H_1 and common for elements of H_2. Then X is common for elements of $H_1 + H_2$, common for elements of $H_1 - H_2$, and common for elements of $H_1\,H_2$.

(39) If X is common for elements of H, then X is common for elements of $|H|$ and common for elements of $-H$.

(40) If X is common for elements of H, then X is common for elements of $r\,H$.

(41) Suppose H_1 is point-convergent on X and H_2 is point-convergent on X. Then

(i) $H_1 + H_2$ is point-convergent on X,

(ii) $\lim_X(H_1 + H_2) = \lim_X H_1 + \lim_X H_2$,

(iii) $H_1 - H_2$ is point-convergent on X,

(iv) $\lim_X(H_1 - H_2) = \lim_X H_1 - \lim_X H_2$,

(v) $H_1\,H_2$ is point-convergent on X, and

(vi) $\lim_X(H_1\,H_2) = \lim_X H_1\,\lim_X H_2$.

(42) Suppose H is point-convergent on X. Then $|H|$ is point-convergent on X and $\lim_X|H| = |\lim_X H|$ and $-H$ is point-convergent on X and $\lim_X(-H) = -\lim_X H$.

(43) If H is point-convergent on X, then $r\,H$ is point-convergent on X and $\lim_X(r\,H) = r\,\lim_X H$.

(44) H is uniform-convergent on X if and only if the following conditions are satisfied:

(i) X is common for elements of H,

(ii) H is point-convergent on X, and

(iii) for every r such that $0 < r$ there exists k such that for all n, x such that $n \geq k$ and $x \in X$ holds $|H(n)(x) - (\lim_X H)(x)| < r$.

In the sequel H is a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$.
One can prove the following proposition

(45) If H is uniform-convergent on X and for every n holds $H(n)$ is continuous on X, then $\lim_X H$ is continuous on X.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[5] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_2.html.

[6] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[7] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rfunct_1.html.

[8] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[9] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/fcont_1.html.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[11] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[12] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[13] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[14] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received May 22, 1992

Published January 2, 2004