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# Real Sequences and Basic Operations on Them

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**Summary.** Definition of real sequence and operations on sequences (multiplication of sequences and multiplication by a real number, addition, subtraction, division and absolute value of sequence) are given.

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The articles [6], [8], [1], [7], [4], [9], [2], [5], [10], and [3] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: $f$ is a function, $n$ is a natural number, $r$, $p$ are real numbers, and $x$ is a set.

A sequence of real numbers is a function from $\mathbb{N}$ into $\mathbb{R}$.

In the sequel $s_1$, $s_2$, $s_3$, $s_4$, $s'_1$, $s'_2$ denote sequences of real numbers.

We now state two propositions:

(3)[1] $f$ is a sequence of real numbers iff $\operatorname{dom} f = \mathbb{N}$ and for every $x$ such that $x \in \mathbb{N}$ holds $f(x)$ is real.

(4) $f$ is a sequence of real numbers iff $\operatorname{dom} f = \mathbb{N}$ and for every $n$ holds $f(n)$ is real.

Let $f$ be a binary relation. We say that $f$ is real-yielding if and only if:

(Def. 1) $\operatorname{rng} f \subseteq \mathbb{R}$.

Let $C$ be a set. One can check that every partial function from $C$ to $\mathbb{R}$ is real-yielding.

One can check that there exists a function which is real-yielding.

Let $f$ be a real-yielding function and let $x$ be a set. One can verify that $f(x)$ is real.

Let $f$ be a real-yielding function and let $x$ be a set. Then $f(x)$ is a real number.

Let $C$ be a set, let $f$ be a partial function from $C$ to $\mathbb{R}$, and let $x$ be a set. Then $f(x)$ is a real number.

Let $f$ be a partial function from $\mathbb{N}$ to $\mathbb{R}$. Let us observe that $f$ is non-empty if and only if:

(Def. 2) $\operatorname{rng} f \subseteq \mathbb{R} \setminus \{0\}$.

We introduce $f$ is non-zero and $f$ is non-zero as synonyms of $f$ is non-empty.

The following four propositions are true:

(6)[2] $s_1$ is non-zero iff for every $x$ such that $x \in \mathbb{N}$ holds $s_1(x) \neq 0$.

(7) $s_1$ is non-zero iff for every $n$ holds $s_1(n) \neq 0$.

[1] The propositions (1) and (2) have been removed.
[2] The proposition (5) has been removed.

(8) For all $s_1$, $s_2$ such that for every $x$ such that $x \in \mathbb{N}$ holds $s_1(x) = s_2(x)$ holds $s_1 = s_2$.

(10)[3] For every $r$ there exists $s_1$ such that $\operatorname{rng} s_1 = \{r\}$.

In this article we present several logical schemes. The scheme *ExRealSeq* deals with a unary functor $\mathcal{F}$ yielding a real number, and states that:

There exists $s_1$ such that for every $n$ holds $s_1(n) = \mathcal{F}(n)$

for all values of the parameter.

The scheme *PartFuncExD'* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a partial function $f$ from $\mathcal{A}$ to $\mathcal{B}$ such that

(i) for every element $d$ of $\mathcal{A}$ holds $d \in \operatorname{dom} f$ iff there exists an element $c$ of $\mathcal{B}$ such that $\mathcal{P}[d,c]$, and

(ii) for every element $d$ of $\mathcal{A}$ such that $d \in \operatorname{dom} f$ holds $\mathcal{P}[d, f(d)]$

for all values of the parameters.

The scheme *LambdaPFD'* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a partial function $f$ from $\mathcal{A}$ to $\mathcal{B}$ such that for every element $d$ of $\mathcal{A}$ holds $d \in \operatorname{dom} f$ iff $\mathcal{P}[d]$ and for every element $d$ of $\mathcal{A}$ such that $d \in \operatorname{dom} f$ holds $f(d) = \mathcal{F}(d)$

for all values of the parameters.

The scheme *UnPartFuncD'* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

Let $f$, $g$ be partial functions from $\mathcal{A}$ to $\mathcal{B}$. Suppose that

(i) $\operatorname{dom} f = \mathcal{C}$,

(ii) for every element $c$ of $\mathcal{A}$ such that $c \in \operatorname{dom} f$ holds $f(c) = \mathcal{F}(c)$,

(iii) $\operatorname{dom} g = \mathcal{C}$, and

(iv) for every element $c$ of $\mathcal{A}$ such that $c \in \operatorname{dom} g$ holds $g(c) = \mathcal{F}(c)$.

Then $f = g$

for all values of the parameters.

Let $C$ be a set and let $f_1$, $f_2$ be partial functions from $C$ to $\mathbb{R}$. The functor $f_1 + f_2$ yielding a partial function from $C$ to $\mathbb{R}$ is defined by:

(Def. 3) $\operatorname{dom}(f_1 + f_2) = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every element $c$ of $C$ such that $c \in \operatorname{dom}(f_1 + f_2)$ holds $(f_1 + f_2)(c) = f_1(c) + f_2(c)$.

Let us observe that the functor $f_1 + f_2$ is commutative. The functor $f_1 - f_2$ yields a partial function from $C$ to $\mathbb{R}$ and is defined as follows:

(Def. 4) $\operatorname{dom}(f_1 - f_2) = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every element $c$ of $C$ such that $c \in \operatorname{dom}(f_1 - f_2)$ holds $(f_1 - f_2)(c) = f_1(c) - f_2(c)$.

The functor $f_1\ f_2$ yielding a partial function from $C$ to $\mathbb{R}$ is defined by:

(Def. 5) $\operatorname{dom}(f_1\ f_2) = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every element $c$ of $C$ such that $c \in \operatorname{dom}(f_1\ f_2)$ holds $(f_1\ f_2)(c) = f_1(c) \cdot f_2(c)$.

Let us observe that the functor $f_1\ f_2$ is commutative.

One can prove the following two propositions:

(11) $s_1 = s_2 + s_3$ iff for every $n$ holds $s_1(n) = s_2(n) + s_3(n)$.

(12) $s_1 = s_2\ s_3$ iff for every $n$ holds $s_1(n) = s_2(n) \cdot s_3(n)$.

Let us consider $s_2$, $s_3$. One can check that $s_2 + s_3$ is total and $s_2\ s_3$ is total.

Let $C$ be a set, let $f$ be a partial function from $C$ to $\mathbb{R}$, and let $r$ be a real number. The functor $r\ f$ yields a partial function from $C$ to $\mathbb{R}$ and is defined as follows:

[3] The proposition (9) has been removed.

(Def. 6) $\mathrm{dom}(r\,f) = \mathrm{dom}\,f$ and for every element $c$ of $C$ such that $c \in \mathrm{dom}(r\,f)$ holds $(r\,f)(c) = r \cdot f(c)$.

Let us consider $r$, $s_1$. Note that $r\,s_1$ is total.
Next we state the proposition

(13) $s_2 = r\,s_3$ iff for every $n$ holds $s_2(n) = r \cdot s_3(n)$.

Let $C$ be a set and let $f$ be a partial function from $C$ to $\mathbb{R}$. The functor $-f$ yielding a partial function from $C$ to $\mathbb{R}$ is defined by:

(Def. 7) $\mathrm{dom}(-f) = \mathrm{dom}\,f$ and for every element $c$ of $C$ such that $c \in \mathrm{dom}(-f)$ holds $(-f)(c) = -f(c)$.

Let us consider $s_1$. Observe that $-s_1$ is total.
We now state the proposition

(14) $s_2 = -s_3$ iff for every $n$ holds $s_2(n) = -s_3(n)$.

Let us consider $s_2$, $s_3$. Note that $s_2 - s_3$ is total.
Next we state the proposition

(15) $s_2 - s_3 = s_2 + -s_3$.

Let us consider $s_1$. The functor ${s_1}^{-1}$ yielding a sequence of real numbers is defined as follows:

(Def. 8) For every $n$ holds ${s_1}^{-1}(n) = s_1(n)^{-1}$.

Let us consider $s_2$, $s_1$. The functor $s_2/s_1$ yields a sequence of real numbers and is defined as follows:

(Def. 9) $s_2/s_1 = s_2\,{s_1}^{-1}$.

Let $C$ be a set and let $f$ be a partial function from $C$ to $\mathbb{R}$. The functor $|f|$ yields a partial function from $C$ to $\mathbb{R}$ and is defined as follows:

(Def. 10) $\mathrm{dom}|f| = \mathrm{dom}\,f$ and for every element $c$ of $C$ such that $c \in \mathrm{dom}|f|$ holds $|f|(c) = |f(c)|$.

Let us consider $s_1$. Note that $|s_1|$ is total.
We now state a number of propositions:

(16) $s_2 = |s_1|$ iff for every $n$ holds $s_2(n) = |s_1(n)|$.

(20)[4] $(s_2 + s_3) + s_4 = s_2 + (s_3 + s_4)$.

(22)[5] $(s_2\,s_3)\,s_4 = s_2\,(s_3\,s_4)$.

(23) $(s_2 + s_3)\,s_4 = s_2\,s_4 + s_3\,s_4$.

(24) $s_4\,(s_2 + s_3) = s_4\,s_2 + s_4\,s_3$.

(25) $-s_1 = (-1)\,s_1$.

(26) $r\,(s_2\,s_3) = (r\,s_2)\,s_3$.

(27) $r\,(s_2\,s_3) = s_2\,(r\,s_3)$.

(28) $(s_2 - s_3)\,s_4 = s_2\,s_4 - s_3\,s_4$.

(29) $s_4\,s_2 - s_4\,s_3 = s_4\,(s_2 - s_3)$.

(30) $r\,(s_2 + s_3) = r\,s_2 + r\,s_3$.

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[4] The propositions (17)–(19) have been removed.
[5] The proposition (21) has been removed.

(31) $(r \cdot p)\, s_1 = r\,(p\, s_1)$.

(32) $r\,(s_2 - s_3) = r\, s_2 - r\, s_3$.

(33) $r\,(s_2/s_1) = (r\, s_2)/s_1$.

(34) $s_2 - (s_3 + s_4) = s_2 - s_3 - s_4$.

(35) $1\, s_1 = s_1$.

(36) $--s_1 = s_1$.

(37) $s_2 - -s_3 = s_2 + s_3$.

(38) $s_2 - (s_3 - s_4) = (s_2 - s_3) + s_4$.

(39) $s_2 + (s_3 - s_4) = (s_2 + s_3) - s_4$.

(40) $(-s_2)\, s_3 = -s_2\, s_3$ and $s_2\, -s_3 = -s_2\, s_3$.

(41) If $s_1$ is non-zero, then ${s_1}^{-1}$ is non-zero.

(42) $({s_1}^{-1})^{-1} = s_1$.

(43) $s_1$ is non-zero and $s_2$ is non-zero iff $s_1\, s_2$ is non-zero.

(44) ${s_1}^{-1}\, {s_2}^{-1} = (s_1\, s_2)^{-1}$.

(45) If $s_1$ is non-zero, then $(s_2/s_1)\, s_1 = s_2$.

(46) $(s'_1/s_1)\,(s'_2/s_2) = (s'_1\, s'_2)/(s_1\, s_2)$.

(47) If $s_1$ is non-zero and $s_2$ is non-zero, then $s_1/s_2$ is non-zero.

(48) $(s_1/s_2)^{-1} = s_2/s_1$.

(49) $s_3\,(s_2/s_1) = (s_3\, s_2)/s_1$.

(50) $s_3/(s_1/s_2) = (s_3\, s_2)/s_1$.

(51) If $s_2$ is non-zero, then $s_3/s_1 = (s_3\, s_2)/(s_1\, s_2)$.

(52) If $r \neq 0$ and $s_1$ is non-zero, then $r\, s_1$ is non-zero.

(53) If $s_1$ is non-zero, then $-s_1$ is non-zero.

(54) $(r\, s_1)^{-1} = r^{-1}\, {s_1}^{-1}$.

(55) $(-s_1)^{-1} = (-1)\, {s_1}^{-1}$.

(56) $-s_2/s_1 = (-s_2)/s_1$ and $s_2/-s_1 = -s_2/s_1$.

(57) $s_2/s_1 + s'_2/s_1 = (s_2 + s'_2)/s_1$ and $s_2/s_1 - s'_2/s_1 = (s_2 - s'_2)/s_1$.

(58) If $s_1$ is non-zero and $s'_1$ is non-zero, then $s_2/s_1 + s'_2/s'_1 = (s_2\, s'_1 + s'_2\, s_1)/(s_1\, s'_1)$ and $s_2/s_1 - s'_2/s'_1 = (s_2\, s'_1 - s'_2\, s_1)/(s_1\, s'_1)$.

(59) $s'_2/s_1/(s'_1/s_2) = (s'_2\, s_2)/(s_1\, s'_1)$.

(60) $|s_1\, s'_1| = |s_1|\, |s'_1|$.

(61) If $s_1$ is non-zero, then $|s_1|$ is non-zero.

(62) $|s_1|^{-1} = |{s_1}^{-1}|$.

(63) $|s'_1/s_1| = |s'_1|/|s_1|$.

(64) $|r\, s_1| = |r|\, |s_1|$.

## References

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[5] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[7] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[8] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[9] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[10] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

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