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Recursive Euclide Algorithm [1]

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Summary. The earlier SCM computer did not contain recursive function, so Trybulec and Nakamura proved the correctness of the Euclid's algorithm only by way of an iterative program. However, the recursive method is a very important programming method, furthermore, for some algorithms, for example Quicksort, only by employing a recursive method (note push-down stack is essentially also a recursive method) can they be implemented. The main goal of the article is to test the recursive function of the SCMPDS computer by proving the correctness of the Euclid's algorithm by way of a recursive program. In this article, we observed that the memory required by the recursive Euclide algorithm is variable but it is still autonomic. Although the algorithm here is more complicated than the non-recursive algorithm, its focus is that the SCMPDS computer will be able to implement many algorithms like Quicksort which the SCM computer cannot do.

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WWW: http://mizar.org/JFM/Vol11/scmp_gcd.html

The articles [15], [3], [13], [2], [4], [10], [11], [12], [8], [7], [5], [1], [6], [14], and [9] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we use the following convention: m, n denote natural numbers, i, j denote instructions of SCMPDS, s denotes a state of SCMPDS, and I, J denote Program-blocks.

The following three propositions are true:

(1) If $m > 0$, then $\gcd(n,m) = \gcd(m, n \bmod m)$.

(2) For all integers i, j such that $i \geq 0$ and $j > 0$ holds $i \gcd j = j \gcd i \bmod j$.

(3) For every natural number m and for every integer j such that $\mathrm{inspos}\, m = j$ holds $\mathrm{inspos}\, m + 2 = 2 \cdot (|j| \div 2) + 4$.

Let k be a natural number. The functor $\mathrm{intpos}\, k$ yields an Int position and is defined by:

(Def. 1) $\mathrm{intpos}\, k = \mathbf{d}_k$.

We now state three propositions:

(4) For all natural numbers n_1, n_2 such that $n_1 \neq n_2$ holds $\mathrm{intpos}\, n_1 \neq \mathrm{intpos}\, n_2$.

(5) For all natural numbers n_1, n_2 holds $\mathrm{DataLoc}(n_1, n_2) = \mathrm{intpos}\, n_1 + n_2$.

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(6) For every state s of SCMPDS and for all natural numbers m_1, m_2 such that $\mathbf{IC}_s = \operatorname{inspos} m_1 + m_2$ holds $\operatorname{ICplusConst}(s, -m_2) = \operatorname{inspos} m_1$.

The Int position GBP is defined as follows:

(Def. 2) $\mathrm{GBP} = \operatorname{intpos} 0$.

The Int position SBP is defined by:

(Def. 3) $\mathrm{SBP} = \operatorname{intpos} 1$.

We now state several propositions:

(7) $\mathrm{GBP} \neq \mathrm{SBP}$.

(8) $\operatorname{card}(I; i) = \operatorname{card} I + 1$.

(9) $\operatorname{card}(i; j) = 2$.

(10) $(I; i)(\operatorname{inspos} \operatorname{card} I) = i$ and $\operatorname{inspos} \operatorname{card} I \in \operatorname{dom}(I; i)$.

(11) $(I; i; J)(\operatorname{inspos} \operatorname{card} I) = i$.

2. The Construction of Recursive Euclide Algorithm

The Program-block GCD-Algorithm is defined by:

(Def. 4) GCD-Algorithm $= (\mathrm{GBP}:=0);\ (\mathrm{SBP}:=7);\ \mathrm{saveIC}(\mathrm{SBP}, \mathrm{RetIC});\ \mathrm{goto}\ 2;\ \mathbf{halt}_{\mathrm{SCMPDS}};\ ((\mathrm{SBP}, 3) <= 0\_\mathrm{goto}\ 9);\ ((\mathrm{SBP}, 6) := (\mathrm{SBP}, 3));\ \mathrm{Divide}(\mathrm{SBP}, 2, \mathrm{SBP}, 3);\ ((\mathrm{SBP}, 7) := (\mathrm{SBP}, 3));\ ((\mathrm{SBP}, 4 + \mathrm{RetSP}) := (\mathrm{GBP}, 1));\ \mathrm{AddTo}(\mathrm{GBP}, 1, 4);\ \mathrm{saveIC}(\mathrm{SBP}, \mathrm{RetIC});\ \mathrm{goto}\ (-7);\ ((\mathrm{SBP}, 2) := (\mathrm{SBP}, 6));\ \mathrm{return}\ \mathrm{SBP}$.

3. The Computation of Recursive Euclide Algorithm

One can prove the following propositions:

(12) $\operatorname{card} \mathrm{GCD\text{-}Algorithm} = 15$.

(13) $n < 15$ iff $\operatorname{inspos} n \in \operatorname{dom} \mathrm{GCD\text{-}Algorithm}$.

(14) $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 0) = \mathrm{GBP}:=0$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 1) = \mathrm{SBP}:=7$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 2) = \mathrm{saveIC}(\mathrm{SBP}, \mathrm{RetIC})$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 3) = \mathrm{goto}\ 2$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 4) = \mathbf{halt}_{\mathrm{SCMPDS}}$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 5) = (\mathrm{SBP}, 3) <= 0\_\mathrm{goto}\ 9$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 6) = (\mathrm{SBP}, 6) := (\mathrm{SBP}, 3)$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 7) = \mathrm{Divide}(\mathrm{SBP}, 2, \mathrm{SBP}, 3)$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 8) = (\mathrm{SBP}, 7) := (\mathrm{SBP}, 3)$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 9) = (\mathrm{SBP}, 4 + \mathrm{RetSP}) := (\mathrm{GBP}, 1)$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 10) = \mathrm{AddTo}(\mathrm{GBP}, 1, 4)$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 11) = \mathrm{saveIC}(\mathrm{SBP}, \mathrm{RetIC})$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 12) = \mathrm{goto}\ (-7)$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 13) = (\mathrm{SBP}, 2) := (\mathrm{SBP}, 6)$ and $(\mathrm{GCD\text{-}Algorithm})(\operatorname{inspos} 14) = \mathrm{return}\ \mathrm{SBP}$.

(15) Let s be a state of SCMPDS. Suppose $\operatorname{Initialized}(\mathrm{GCD\text{-}Algorithm}) \subseteq s$. Then $\mathbf{IC}_{(\operatorname{Computation}(s))(4)} = \operatorname{inspos} 5$ and $(\operatorname{Computation}(s))(4)(\mathrm{GBP}) = 0$ and $(\operatorname{Computation}(s))(4)(\mathrm{SBP}) = 7$ and $(\operatorname{Computation}(s))(4)(\operatorname{intpos} 7 + \mathrm{RetIC}) = \operatorname{inspos} 2$ and $(\operatorname{Computation}(s))(4)(\operatorname{intpos} 9) = s(\operatorname{intpos} 9)$ and $(\operatorname{Computation}(s))(4)(\operatorname{intpos} 10) = s(\operatorname{intpos} 10)$.

(16) Let s be a state of SCMPDS. Suppose $\mathrm{GCD\text{-}Algorithm} \subseteq s$ and $\mathbf{IC}_s = \operatorname{inspos} 5$ and $s(\mathrm{SBP}) > 0$ and $s(\mathrm{GBP}) = 0$ and $s(\operatorname{DataLoc}(s(\mathrm{SBP}), 3)) \geq 0$ and $s(\operatorname{DataLoc}(s(\mathrm{SBP}), 2)) \geq s(\operatorname{DataLoc}(s(\mathrm{SBP}), 3))$. Then there exists n such that

(i) $\operatorname{CurInstr}((\operatorname{Computation}(s))(n)) = \mathrm{return}\ \mathrm{SBP}$,

(ii) $s(\mathrm{SBP}) = (\operatorname{Computation}(s))(n)(\mathrm{SBP})$,

(iii) $(\mathrm{Computation}(s))(n)(\mathrm{DataLoc}(s(\mathrm{SBP}),2)) = s(\mathrm{DataLoc}(s(\mathrm{SBP}),2)) \gcd s(\mathrm{DataLoc}(s(\mathrm{SBP}),3))$, and

(iv) for every natural number j such that $1 < j$ and $j \leq s(\mathrm{SBP}) + 1$ holds $s(\mathrm{intpos}\, j) = (\mathrm{Computation}(s))(n)(\mathrm{intpos}\, j)$.

(17) Let s be a state of SCMPDS. Suppose GCD-Algorithm $\subseteq s$ and $\mathbf{IC}_s = \mathrm{inspos}\, 5$ and $s(\mathrm{SBP}) > 0$ and $s(\mathrm{GBP}) = 0$ and $s(\mathrm{DataLoc}(s(\mathrm{SBP}),3)) \geq 0$ and $s(\mathrm{DataLoc}(s(\mathrm{SBP}),2)) \geq 0$. Then there exists n such that

(i) $\mathrm{CurInstr}((\mathrm{Computation}(s))(n)) = \mathrm{return}\, \mathrm{SBP}$,

(ii) $s(\mathrm{SBP}) = (\mathrm{Computation}(s))(n)(\mathrm{SBP})$,

(iii) $(\mathrm{Computation}(s))(n)(\mathrm{DataLoc}(s(\mathrm{SBP}),2)) = s(\mathrm{DataLoc}(s(\mathrm{SBP}),2)) \gcd s(\mathrm{DataLoc}(s(\mathrm{SBP}),3))$, and

(iv) for every natural number j such that $1 < j$ and $j \leq s(\mathrm{SBP}) + 1$ holds $s(\mathrm{intpos}\, j) = (\mathrm{Computation}(s))(n)(\mathrm{intpos}\, j)$.

4. The Correctness of Recursive Euclide Algorithm

One can prove the following proposition

(18) Let s be a state of SCMPDS. Suppose $\mathrm{Initialized}(\text{GCD-Algorithm}) \subseteq s$. Let x, y be integers. If $s(\mathrm{intpos}\, 9) = x$ and $s(\mathrm{intpos}\, 10) = y$ and $x \geq 0$ and $y \geq 0$, then $(\mathrm{Result}(s))(\mathrm{intpos}\, 9) = x \gcd y$.

5. The Autonomy of Recursive Euclide Algorithm

One can prove the following proposition

(19) Let p be a finite partial state of SCMPDS and x, y be integers. If $y \geq 0$ and $x \geq y$ and $p = [\mathrm{intpos}\, 9 \longmapsto x, \mathrm{intpos}\, 10 \longmapsto y]$, then $\mathrm{Initialized}(\text{GCD-Algorithm}) +\cdot p$ is autonomic.

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