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Subspaces and Cosets of Subspace of Real Unitary Space

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Summary. In this article, subspace and the coset of subspace of real unitary space are defined. And we discuss some of their fundamental properties.

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The articles [6], [3], [11], [8], [7], [1], [12], [2], [5], [10], [9], and [4] provide the notation and terminology for this paper.

1. Definition and Axioms of the Subspace of Real Unitary Space

Let V be a real unitary space. A real unitary space is said to be a subspace of V if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the zero of it $=$ the zero of V,

(iii) the addition of it $=$ (the addition of V) $\restriction$ [: the carrier of it, the carrier of it :],

(iv) the external multiplication of it $=$ (the external multiplication of V) $\restriction$ [: $\mathbb{R}$, the carrier of it :], and

(v) the scalar product of it $=$ (the scalar product of V) $\restriction$ [: the carrier of it, the carrier of it :].

We now state a number of propositions:

(1) Let V be a real unitary space, W_1, W_2 be subspaces of V, and x be a set. If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(2) For every real unitary space V and for every subspace W of V and for every set x such that $x \in W$ holds $x \in V$.

(3) For every real unitary space V and for every subspace W of V holds every vector of W is a vector of V.

(4) For every real unitary space V and for every subspace W of V holds $0_W = 0_V$.

(5) For every real unitary space V and for all subspaces W_1, W_2 of V holds $0_{(W_1)} = 0_{(W_2)}$.

(6) Let V be a real unitary space, W be a subspace of V, u, v be vectors of V, and w_1, w_2 be vectors of W. If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(7) Let V be a real unitary space, W be a subspace of V, v be a vector of V, w be a vector of W, and a be a real number. If $w = v$, then $a \cdot w = a \cdot v$.

(8) Let V be a real unitary space, W be a subspace of V, v_1, v_2 be vectors of V, and w_1, w_2 be vectors of W. If $w_1 = v_1$ and $w_2 = v_2$, then $(w_1|w_2) = (v_1|v_2)$.

(9) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and w be a vector of W. If $w = v$, then $-v = -w$.

(10) Let V be a real unitary space, W be a subspace of V, u, v be vectors of V, and w_1, w_2 be vectors of W. If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(11) For every real unitary space V and for every subspace W of V holds $0_V \in W$.

(12) For every real unitary space V and for all subspaces W_1, W_2 of V holds $0_{(W_1)} \in W_2$.

(13) For every real unitary space V and for every subspace W of V holds $0_W \in V$.

(14) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $u \in W$ and $v \in W$, then $u + v \in W$.

(15) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $v \in W$, then $a \cdot v \in W$.

(16) For every real unitary space V and for every subspace W of V and for every vector v of V such that $v \in W$ holds $-v \in W$.

(17) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $u \in W$ and $v \in W$, then $u - v \in W$.

(18) Let V be a real unitary space, V_1 be a subset of V, D be a non empty set, d_1 be an element of D, A be a binary operation on D, M be a function from $[:\mathbb{R}, D:]$ into D, and S be a function from $[:D, D:]$ into $\mathbb{R}$. Suppose that

(i) $V_1 = D$,

(ii) $d_1 = 0_V$,

(iii) $A = (\text{the addition of } V) \restriction [:V_1, V_1:]$,

(iv) $M = (\text{the external multiplication of } V) \restriction [:\mathbb{R}, V_1:]$, and

(v) $S = (\text{the scalar product of } V) \restriction [:V_1, V_1:]$.

Then $\langle D, d_1, A, M, S \rangle$ is a subspace of V.

(19) Every real unitary space V is a subspace of V.

(20) For all strict real unitary spaces V, X such that V is a subspace of X and X is a subspace of V holds $V = X$.

(21) Let V, X, Y be real unitary spaces. Suppose V is a subspace of X and X is a subspace of Y. Then V is a subspace of Y.

(22) Let V be a real unitary space and W_1, W_2 be subspaces of V. Suppose the carrier of $W_1 \subseteq$ the carrier of W_2. Then W_1 is a subspace of W_2.

(23) Let V be a real unitary space and W_1, W_2 be subspaces of V. Suppose that for every vector v of V such that $v \in W_1$ holds $v \in W_2$. Then W_1 is a subspace of W_2.

Let V be a real unitary space. Note that there exists a subspace of V which is strict.

Next we state several propositions:

(24) Let V be a real unitary space and W_1, W_2 be strict subspaces of V. If the carrier of $W_1 =$ the carrier of W_2, then $W_1 = W_2$.

(25) Let V be a real unitary space and W_1, W_2 be strict subspaces of V. If for every vector v of V holds $v \in W_1$ iff $v \in W_2$, then $W_1 = W_2$.

(26) Let V be a strict real unitary space and W be a strict subspace of V. If the carrier of $W =$ the carrier of V, then $W = V$.

(27) Let V be a strict real unitary space and W be a strict subspace of V. If for every vector v of V holds $v \in W$ iff $v \in V$, then $W = V$.

(28) Let V be a real unitary space, W be a subspace of V, and V_1 be a subset of V. If the carrier of $W = V_1$, then V_1 is linearly closed.

(29) Let V be a real unitary space, W be a subspace of V, and V_1 be a subset of V. Suppose $V_1 \neq \emptyset$ and V_1 is linearly closed. Then there exists a strict subspace W of V such that $V_1 =$ the carrier of W.

2. Definition of Zero Subspace and Improper Subspace of Real Unitary Space

Let V be a real unitary space. The functor $\mathbf{0}_V$ yields a strict subspace of V and is defined as follows:

(Def. 2) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let V be a real unitary space. The functor Ω_V yielding a strict subspace of V is defined as follows:

(Def. 3) $\Omega_V =$ the unitary space structure of V.

3. Theorems of Zero Subspace and Improper Subspace

The following propositions are true:

(30) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_W = \mathbf{0}_V$.

(31) For every real unitary space V and for all subspaces W_1, W_2 of V holds $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(32) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_W$ is a subspace of V.

(33) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_V$ is a subspace of W.

(34) For every real unitary space V and for all subspaces W_1, W_2 of V holds $\mathbf{0}_{(W_1)}$ is a subspace of W_2.

(35) Every strict real unitary space V is a subspace of Ω_V.

4. The Cosets of Subspace of Real Unitary Space

Let V be a real unitary space, let v be a vector of V, and let W be a subspace of V. The functor $v+W$ yielding a subset of V is defined by:

(Def. 4) $v+W = \{v+u; u$ ranges over vectors of V: $u \in W\}$.

Let V be a real unitary space and let W be a subspace of V. A subset of V is called a coset of W if:

(Def. 5) There exists a vector v of V such that it $= v+W$.

5. Theorems of the Cosets

One can prove the following propositions:

(36) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $0_V \in v+W$ if and only if $v \in W$.

(37) For every real unitary space V and for every subspace W of V and for every vector v of V holds $v \in v+W$.

(38) For every real unitary space V and for every subspace W of V holds $0_V + W =$ the carrier of W.

(39) For every real unitary space V and for every vector v of V holds $v + \mathbf{0}_V = \{v\}$.

(40) For every real unitary space V and for every vector v of V holds $v + \Omega_V =$ the carrier of V.

(41) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $0_V \in v+W$ if and only if $v+W =$ the carrier of W.

(42) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $v \in W$ if and only if $v+W =$ the carrier of W.

(43) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $v \in W$, then $a \cdot v + W =$ the carrier of W.

(44) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $a \neq 0$ and $a \cdot v + W =$ the carrier of W, then $v \in W$.

(45) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $v \in W$ if and only if $-v+W =$ the carrier of W.

(46) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in W$ if and only if $v+W = v+u+W$.

(47) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in W$ if and only if $v+W = (v-u)+W$.

(48) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $v \in u+W$ if and only if $u+W = v+W$.

(49) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $v+W = -v+W$ if and only if $v \in W$.

(50) Let V be a real unitary space, W be a subspace of V, and u, v_1, v_2 be vectors of V. If $u \in v_1+W$ and $u \in v_2+W$, then $v_1+W = v_2+W$.

(51) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $u \in v+W$ and $u \in -v+W$, then $v \in W$.

(52) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $a \neq 1$ and $a \cdot v \in v+W$, then $v \in W$.

(53) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $v \in W$, then $a \cdot v \in v+W$.

(54) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $-v \in v+W$ if and only if $v \in W$.

(55) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u+v \in v+W$ if and only if $u \in W$.

(56) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $v-u \in v+W$ if and only if $u \in W$.

(57) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in v+W$ if and only if there exists a vector v_1 of V such that $v_1 \in W$ and $u = v+v_1$.

(58) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in v+W$ if and only if there exists a vector v_1 of V such that $v_1 \in W$ and $u = v-v_1$.

(59) Let V be a real unitary space, W be a subspace of V, and v_1, v_2 be vectors of V. Then there exists a vector v of V such that $v_1 \in v+W$ and $v_2 \in v+W$ if and only if $v_1 - v_2 \in W$.

(60) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $v+W = u+W$, then there exists a vector v_1 of V such that $v_1 \in W$ and $v+v_1 = u$.

(61) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $v+W = u+W$, then there exists a vector v_1 of V such that $v_1 \in W$ and $v-v_1 = u$.

(62) Let V be a real unitary space, W_1, W_2 be strict subspaces of V, and v be a vector of V. Then $v+W_1 = v+W_2$ if and only if $W_1 = W_2$.

(63) Let V be a real unitary space, W_1, W_2 be strict subspaces of V, and u, v be vectors of V. If $v+W_1 = u+W_2$, then $W_1 = W_2$.

(64) Let V be a real unitary space, W be a subspace of V, and C be a coset of W. Then C is linearly closed if and only if $C =$ the carrier of W.

(65) Let V be a real unitary space, W_1, W_2 be strict subspaces of V, C_1 be a coset of W_1, and C_2 be a coset of W_2. If $C_1 = C_2$, then $W_1 = W_2$.

(66) Let V be a real unitary space, W be a subspace of V, C be a coset of W, and v be a vector of V. Then $\{v\}$ is a coset of $\mathbf{0}_V$.

(67) Let V be a real unitary space, W be a subspace of V, V_1 be a subset of V, and v be a vector of V. If V_1 is a coset of $\mathbf{0}_V$, then there exists a vector v of V such that $V_1 = \{v\}$.

(68) For every real unitary space V and for every subspace W of V holds the carrier of W is a coset of W.

(69) For every real unitary space V holds the carrier of V is a coset of Ω_V.

(70) Let V be a real unitary space, W be a subspace of V, and V_1 be a subset of V. If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of V.

(71) Let V be a real unitary space, W be a subspace of V, and C be a coset of W. Then $0_V \in C$ if and only if $C =$ the carrier of W.

(72) Let V be a real unitary space, W be a subspace of V, C be a coset of W, and u be a vector of V. Then $u \in C$ if and only if $C = u+W$.

(73) Let V be a real unitary space, W be a subspace of V, C be a coset of W, and u, v be vectors of V. If $u \in C$ and $v \in C$, then there exists a vector v_1 of V such that $v_1 \in W$ and $u+v_1 = v$.

(74) Let V be a real unitary space, W be a subspace of V, C be a coset of W, and u, v be vectors of V. If $u \in C$ and $v \in C$, then there exists a vector v_1 of V such that $v_1 \in W$ and $u-v_1 = v$.

(75) Let V be a real unitary space, W be a subspace of V, and v_1, v_2 be vectors of V. Then there exists a coset C of W such that $v_1 \in C$ and $v_2 \in C$ if and only if $v_1 - v_2 \in W$.

(76) Let V be a real unitary space, W be a subspace of V, u be a vector of V, and B, C be cosets of W. If $u \in B$ and $u \in C$, then $B = C$.

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