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Real Linear Space of Real Sequences

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Summary. The article is a continuation of [15]. As the example of real linear spaces, we introduce the arithmetic addition in the set of real sequences and also introduce the multiplication. This set has the arithmetic structure which depends on these arithmetic operations.

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The articles [12], [16], [6], [1], [13], [7], [17], [3], [5], [4], [15], [14], [10], [9], [8], [11], and [2] provide the notation and terminology for this paper.

The non empty set the set of real sequences is defined by:

(Def. 1) For every set x holds $x \in$ the set of real sequences iff x is a sequence of real numbers.

Let a be a set. Let us assume that $a \in$ the set of real sequences. The functor $\mathrm{id}_{\mathrm{seq}}(a)$ yielding a sequence of real numbers is defined by:

(Def. 2) $\mathrm{id}_{\mathrm{seq}}(a) = a$.

Let a be a set. Let us assume that $a \in \mathbb{R}$. The functor $\mathrm{id}_{\mathbb{R}}(a)$ yields a real number and is defined as follows:

(Def. 3) $\mathrm{id}_{\mathbb{R}}(a) = a$.

Next we state two propositions:

(1) There exists a binary operation A_1 on the set of real sequences such that for all elements a, b of the set of real sequences holds $A_1(a, b) = \mathrm{id}_{\mathrm{seq}}(a) + \mathrm{id}_{\mathrm{seq}}(b)$ and A_1 is commutative and associative.

(2) There exists a function f from $[:\mathbb{R}, \text{the set of real sequences}:]$ into the set of real sequences such that for all sets r, x if $r \in \mathbb{R}$ and $x \in$ the set of real sequences, then $f(\langle r, x\rangle) = \mathrm{id}_{\mathbb{R}}(r)\ \mathrm{id}_{\mathrm{seq}}(x)$.

The binary operation $\mathrm{add}_{\mathrm{seq}}$ on the set of real sequences is defined by:

(Def. 4) For all elements a, b of the set of real sequences holds $\mathrm{add}_{\mathrm{seq}}(a, b) = \mathrm{id}_{\mathrm{seq}}(a) + \mathrm{id}_{\mathrm{seq}}(b)$.

The function $\mathrm{mult}_{\mathrm{seq}}$ from $[:\mathbb{R}, \text{the set of real sequences}:]$ into the set of real sequences is defined by:

(Def. 5) For all sets r, x such that $r \in \mathbb{R}$ and $x \in$ the set of real sequences holds $\mathrm{mult}_{\mathrm{seq}}(\langle r, x\rangle) = \mathrm{id}_{\mathbb{R}}(r)\ \mathrm{id}_{\mathrm{seq}}(x)$.

The element Zeroseq of the set of real sequences is defined by:

(Def. 6) For every natural number n holds $(\mathrm{id}_{\mathrm{seq}}(\mathrm{Zeroseq}))(n) = 0$.

The following propositions are true:

(3) For every sequence x of real numbers holds $\mathrm{id}_{\mathrm{seq}}(x) = x$.

(4) For all vectors v, w of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $v + w = \mathrm{id}_{\mathrm{seq}}(v) + \mathrm{id}_{\mathrm{seq}}(w)$.

(5) For every real number r and for every vector v of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $r \cdot v = r\ \mathrm{id}_{\mathrm{seq}}(v)$.

Let us mention that $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ is Abelian.
The following propositions are true:

(6) For all vectors u, v, w of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(u + v) + w = u + (v + w)$.

(7) For every vector v of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $v + 0_{\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle} = v$.

(8) Let v be a vector of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$. Then there exists a vector w of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ such that $v + w = 0_{\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle}$.

(9) For every real number a and for all vectors v, w of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $a \cdot (v + w) = a \cdot v + a \cdot w$.

(10) For all real numbers a, b and for every vector v of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(a + b) \cdot v = a \cdot v + b \cdot v$.

(11) For all real numbers a, b and for every vector v of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$.

(12) For every vector v of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $1 \cdot v = v$.

The real linear space the linear space of real sequences is defined by:

(Def. 7) The linear space of real sequences $=$ $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$.

Let X be a real linear space and let X_1 be a subset of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Add\_}(X_1, X)$ yields a binary operation on X_1 and is defined as follows:

(Def. 8) $\mathrm{Add\_}(X_1, X) = (\text{the addition of } X) \upharpoonright [: X_1, X_1 :]$.

Let X be a real linear space and let X_1 be a subset of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Mult\_}(X_1, X)$ yields a function from $[: \mathbb{R}, X_1 :]$ into X_1 and is defined by:

(Def. 9) $\mathrm{Mult\_}(X_1, X) = (\text{the external multiplication of } X) \upharpoonright [: \mathbb{R}, X_1 :]$.

Let X be a real linear space and let X_1 be a subset of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Zero\_}(X_1, X)$ yields an element of X_1 and is defined by:

(Def. 10) $\mathrm{Zero\_}(X_1, X) = 0_X$.

The following proposition is true

(13) Let V be a real linear space and V_1 be a subset of V. Suppose V_1 is linearly closed and non empty. Then $\langle V_1, \mathrm{Zero\_}(V_1, V), \mathrm{Add\_}(V_1, V), \mathrm{Mult\_}(V_1, V)\rangle$ is a subspace of V.

The subset the set of l2-real sequences of the linear space of real sequences is defined by the conditions (Def. 11).

(Def. 11)(i) The set of l2-real sequences is non empty, and

(ii) for every set x holds $x \in$ the set of l2-real sequences iff $x \in$ the set of real sequences and $\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(x)$ is summable.

One can prove the following propositions:

(14) The set of l2-real sequences is linearly closed and the set of l2-real sequences is non empty.

(15) $\langle$the set of l2-real sequences, Zero_(the set of l2-real sequences, the linear space of real sequences), Add_(the set of l2-real sequences, the linear space of real sequences), Mult_(the set of l2-real sequences, the linear space of real sequences)$\rangle$ is a subspace of the linear space of real sequences.

(16) $\langle$the set of l2-real sequences, Zero_(the set of l2-real sequences, the linear space of real sequences), Add_(the set of l2-real sequences, the linear space of real sequences), Mult_(the set of l2-real sequences, the linear space of real sequences)$\rangle$ is a real linear space.

(17)(i) The carrier of the linear space of real sequences $=$ the set of real sequences,

(ii) for every set x holds x is an element of the linear space of real sequences iff x is a sequence of real numbers,

(iii) for every set x holds x is a vector of the linear space of real sequences iff x is a sequence of real numbers,

(iv) for every vector u of the linear space of real sequences holds $u = \mathrm{id}_{\mathrm{seq}}(u)$,

(v) for all vectors u, v of the linear space of real sequences holds $u + v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$, and

(vi) for every real number r and for every vector u of the linear space of real sequences holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$.

(18) There exists a function f from [:the set of l2-real sequences, the set of l2-real sequences:] into $\mathbb{R}$ such that for all sets x, y if $x \in$ the set of l2-real sequences and $y \in$ the set of l2-real sequences, then $f(\langle x, y\rangle) = \sum(\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(y))$.

The function $\mathrm{scalar}_{\mathrm{seq}}$ from [:the set of l2-real sequences, the set of l2-real sequences:] into $\mathbb{R}$ is defined by the condition (Def. 12).

(Def. 12) Let x, y be sets. Suppose $x \in$ the set of l2-real sequences and $y \in$ the set of l2-real sequences. Then $\mathrm{scalar}_{\mathrm{seq}}(\langle x, y\rangle) = \sum(\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(y))$.

Let us note that $\langle$the set of l2-real sequences, Zero_(the set of l2-real sequences, the linear space of real sequences), Add_(the set of l2-real sequences, the linear space of real sequences), Mult_(the set of l2-real sequences, the linear space of real sequences), $\mathrm{scalar}_{\mathrm{seq}}\rangle$ is non empty.

The non empty unitary space structure l2-Space is defined by the condition (Def. 13).

(Def. 13) l2-Space $=$ $\langle$the set of l2-real sequences, Zero_(the set of l2-real sequences, the linear space of real sequences), Add_(the set of l2-real sequences, the linear space of real sequences), Mult_(the set of l2-real sequences, the linear space of real sequences), $\mathrm{scalar}_{\mathrm{seq}}\rangle$.

One can prove the following propositions:

(19) Let l be a unitary space structure. Suppose $\langle$the carrier of l, the zero of l, the addition of l, the external multiplication of $l\rangle$ is a real linear space. Then l is a real linear space.

(20) Let r_1 be a sequence of real numbers. If for every natural number n holds $r_1(n) = 0$, then r_1 is summable and $\sum r_1 = 0$.

(21) Let r_1 be a sequence of real numbers. Suppose for every natural number n holds $0 \leq r_1(n)$ and r_1 is summable and $\sum r_1 = 0$. Let n be a natural number. Then $r_1(n) = 0$.

Let us note that l2-Space is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

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