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Introduction to Probability

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Summary. Definitions of Elementary Event and Event in any sample space E are given. Next, the probability of an Event when E is finite is introduced and some properties of this function are investigated. Last part of the paper is devoted to the conditional probability and essential properties of this function (Bayes Theorem).

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The articles [8], [9], [4], [7], [6], [2], [5], [1], and [3] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: E denotes a non empty set, a denotes an element of E, A, B denote subsets of E, Y denotes a set, and p denotes a finite sequence.

Let E be a non empty set. Observe that there exists a subset of E which is non empty and trivial.

Let us consider E. An elementary event of E is a non empty trivial subset of E.

Next we state the proposition

(1) Let e be a non empty subset of E. Then e is an elementary event of E if and only if for every Y holds $Y \subseteq e$ iff $Y = \emptyset$ or $Y = e$.

Let us consider E. One can check that every elementary event of E is finite.

In the sequel e, e_1, e_2 are elementary events of E.

Next we state several propositions:

(5)[1] If $e = A \cup B$ and $A \neq B$, then $A = \emptyset$ and $B = e$ or $A = e$ and $B = \emptyset$.

(6) If $e = A \cup B$, then $A = e$ and $B = e$ or $A = e$ and $B = \emptyset$ or $A = \emptyset$ and $B = e$.

(7) $\{a\}$ is an elementary event of E.

(10)[2] If $e_1 \subseteq e_2$, then $e_1 = e_2$.

(11) There exists a such that $a \in E$ and $e = \{a\}$.

(12) There exists e which is an elementary event of E.

(14)[3] There exists p such that p is a finite sequence of elements of E and $\operatorname{rng} p = e$ and $\operatorname{len} p = 1$.

Let E be a set. An event of E is a subset of E.

Next we state several propositions:

[1] The propositions (2)–(4) have been removed.
[2] The propositions (8) and (9) have been removed.
[3] The proposition (13) has been removed.

(22)[4] Let E be a non empty set, e be an elementary event of E, and A be an event of E. Then e misses A or $e \cap A = e$.

(25)[5] For every non empty set E and for every event A of E such that $A \neq \emptyset$ there exists an elementary event e of E such that $e \subseteq A$.

(26) Let E be a non empty set, e be an elementary event of E, and A be an event of E. If $e \subseteq A \cup A^{\rm c}$, then $e \subseteq A$ or $e \subseteq A^{\rm c}$.

(27) $e_1 = e_2$ or e_1 misses e_2.

(34)[6] $A \cap B$ misses $A \cap B^{\rm c}$.

Let E be a finite non empty set and let A be an event of E. The functor $\mathrm{P}(A)$ yields a real number and is defined as follows:

(Def. 4)[7] $\mathrm{P}(A) = \frac{\mathrm{card}\,A}{\mathrm{card}\,E}$.

We now state a number of propositions:

(38)[8] For every finite non empty set E and for every elementary event e of E holds $\mathrm{P}(e) = \frac{1}{\mathrm{card}\,E}$.

(39) For every finite non empty set E holds $\mathrm{P}(\Omega_E) = 1$.

(40) For every finite non empty set E holds $\mathrm{P}(\emptyset_E) = 0$.

(41) For every finite non empty set E and for all events A, B of E such that A misses B holds $\mathrm{P}(A \cap B) = 0$.

(42) For every finite non empty set E and for every event A of E holds $\mathrm{P}(A) \leq 1$.

(43) For every finite non empty set E and for every event A of E holds $0 \leq \mathrm{P}(A)$.

(44) For every finite non empty set E and for all events A, B of E such that $A \subseteq B$ holds $\mathrm{P}(A) \leq \mathrm{P}(B)$.

(46)[9] For every finite non empty set E and for all events A, B of E holds $\mathrm{P}(A \cup B) = (\mathrm{P}(A) + \mathrm{P}(B)) - \mathrm{P}(A \cap B)$.

(47) For every finite non empty set E and for all events A, B of E such that A misses B holds $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

(48) For every finite non empty set E and for every event A of E holds $\mathrm{P}(A) = 1 - \mathrm{P}(A^{\rm c})$ and $\mathrm{P}(A^{\rm c}) = 1 - \mathrm{P}(A)$.

(49) For every finite non empty set E and for all events A, B of E holds $\mathrm{P}(A \setminus B) = \mathrm{P}(A) - \mathrm{P}(A \cap B)$.

(50) For every finite non empty set E and for all events A, B of E such that $B \subseteq A$ holds $\mathrm{P}(A \setminus B) = \mathrm{P}(A) - \mathrm{P}(B)$.

(51) For every finite non empty set E and for all events A, B of E holds $\mathrm{P}(A \cup B) \leq \mathrm{P}(A) + \mathrm{P}(B)$.

(53)[10] For every finite non empty set E and for all events A, B of E holds $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^{\rm c})$.

[4] The propositions (15)–(21) have been removed.
[5] The propositions (23) and (24) have been removed.
[6] The propositions (28)–(33) have been removed.
[7] The definitions (Def. 1)–(Def. 3) have been removed.
[8] The propositions (35)–(37) have been removed.
[9] The proposition (45) has been removed.
[10] The proposition (52) has been removed.

(54) For every finite non empty set E and for all events A, B of E holds $\mathrm{P}(A) = \mathrm{P}(A \cup B) - \mathrm{P}(B \setminus A)$.

(55) For every finite non empty set E and for all events A, B of E holds $\mathrm{P}(A) + \mathrm{P}(A^{\rm c} \cap B) = \mathrm{P}(B) + \mathrm{P}(B^{\rm c} \cap A)$.

(56) For every finite non empty set E and for all events A, B, C of E holds $\mathrm{P}(A \cup B \cup C) = ((\mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C)) - (\mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) + \mathrm{P}(B \cap C))) + \mathrm{P}(A \cap B \cap C)$.

(57) Let E be a finite non empty set and A, B, C be events of E. Suppose A misses B and A misses C and B misses C. Then $\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C)$.

(58) For every finite non empty set E and for all events A, B of E holds $\mathrm{P}(A) - \mathrm{P}(B) \leq \mathrm{P}(A \setminus B)$.

Let E be a finite non empty set and let B, A be events of E. The functor $\mathrm{P}(A/B)$ yielding a real number is defined by:

(Def. 5) $\mathrm{P}(A/B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.

The following propositions are true:

(60)[11] For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ holds $\mathrm{P}(A \cap B) = \mathrm{P}(A/B) \cdot \mathrm{P}(B)$.

(61) For every finite non empty set E and for every event A of E holds $\mathrm{P}(A/\Omega_E) = \mathrm{P}(A)$.

(62) For every finite non empty set E holds $\mathrm{P}(\Omega_E/\Omega_E) = 1$.

(63) For every finite non empty set E holds $\mathrm{P}(\emptyset_E/\Omega_E) = 0$.

(64) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ holds $\mathrm{P}(A/B) \leq 1$.

(65) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ holds $0 \leq \mathrm{P}(A/B)$.

(66) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ holds $\mathrm{P}(A/B) = 1 - \frac{\mathrm{P}(B \setminus A)}{\mathrm{P}(B)}$.

(67) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ and $A \subseteq B$ holds $\mathrm{P}(A/B) = \frac{\mathrm{P}(A)}{\mathrm{P}(B)}$.

(68) For every finite non empty set E and for all events A, B of E such that A misses B holds $\mathrm{P}(A/B) = 0$.

(69) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(A)$ and $0 < \mathrm{P}(B)$ holds $\mathrm{P}(A) \cdot \mathrm{P}(B/A) = \mathrm{P}(B) \cdot \mathrm{P}(A/B)$.

(70) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ holds $\mathrm{P}(A/B) = 1 - \mathrm{P}(A^{\rm c}/B)$ and $\mathrm{P}(A^{\rm c}/B) = 1 - \mathrm{P}(A/B)$.

(71) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ and $B \subseteq A$ holds $\mathrm{P}(A/B) = 1$.

(72) For every finite non empty set E and for every event B of E such that $0 < \mathrm{P}(B)$ holds $\mathrm{P}(\Omega_E/B) = 1$.

(73) For every finite non empty set E and for every event A of E such that $0 < \mathrm{P}(A)$ holds $\mathrm{P}(A^{\rm c}/A) = 0$.

[11] The proposition (59) has been removed.

(74) For every finite non empty set E and for every event A of E such that $\mathrm{P}(A) < 1$ holds $\mathrm{P}(A/A^{\mathrm{c}}) = 0$.

(75) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ and A misses B holds $\mathrm{P}(A^{\mathrm{c}}/B) = 1$.

(76) Let E be a finite non empty set and A, B be events of E. If $0 < \mathrm{P}(A)$ and $\mathrm{P}(B) < 1$ and A misses B, then $\mathrm{P}(A/B^{\mathrm{c}}) = \frac{\mathrm{P}(A)}{1-\mathrm{P}(B)}$.

(77) Let E be a finite non empty set and A, B be events of E. If $0 < \mathrm{P}(A)$ and $\mathrm{P}(B) < 1$ and A misses B, then $\mathrm{P}(A^{\mathrm{c}}/B^{\mathrm{c}}) = 1 - \frac{\mathrm{P}(A)}{1-\mathrm{P}(B)}$.

(78) For every finite non empty set E and for all events A, B, C of E such that $0 < \mathrm{P}(B \cap C)$ and $0 < \mathrm{P}(C)$ holds $\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A/(B \cap C)) \cdot \mathrm{P}(B/C) \cdot \mathrm{P}(C)$.

(79) For every finite non empty set E and for all events A, B of E such that $0 < \mathrm{P}(B)$ and $\mathrm{P}(B) < 1$ holds $\mathrm{P}(A) = \mathrm{P}(A/B) \cdot \mathrm{P}(B) + \mathrm{P}(A/B^{\mathrm{c}}) \cdot \mathrm{P}(B^{\mathrm{c}})$.

(80) Let E be a finite non empty set and A, B_1, B_2 be events of E. If $0 < \mathrm{P}(B_1)$ and $0 < \mathrm{P}(B_2)$ and $B_1 \cup B_2 = E$ and B_1 misses B_2, then $\mathrm{P}(A) = \mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2)$.

(81) Let E be a finite non empty set and A, B_1, B_2, B_3 be events of E. Suppose $0 < \mathrm{P}(B_1)$ and $0 < \mathrm{P}(B_2)$ and $0 < \mathrm{P}(B_3)$ and $B_1 \cup B_2 \cup B_3 = E$ and B_1 misses B_2 and B_1 misses B_3 and B_2 misses B_3. Then $\mathrm{P}(A) = \mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2) + \mathrm{P}(A/B_3) \cdot \mathrm{P}(B_3)$.

(82) Let E be a finite non empty set and A, B_1, B_2 be events of E. Suppose $0 < \mathrm{P}(B_1)$ and $0 < \mathrm{P}(B_2)$ and $B_1 \cup B_2 = E$ and B_1 misses B_2. Then $\mathrm{P}(B_1/A) = \frac{\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1)}{\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2)}$.

(83) Let E be a finite non empty set and A, B_1, B_2, B_3 be events of E. Suppose $0 < \mathrm{P}(B_1)$ and $0 < \mathrm{P}(B_2)$ and $0 < \mathrm{P}(B_3)$ and $B_1 \cup B_2 \cup B_3 = E$ and B_1 misses B_2 and B_1 misses B_3 and B_2 misses B_3. Then $\mathrm{P}(B_1/A) = \frac{\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1)}{\mathrm{P}(A/B_1) \cdot \mathrm{P}(B_1) + \mathrm{P}(A/B_2) \cdot \mathrm{P}(B_2) + \mathrm{P}(A/B_3) \cdot \mathrm{P}(B_3)}$.

Let E be a finite non empty set and let A, B be events of E. We say that A and B are independent if and only if:

(Def. 6) $\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$.

Let us note that the predicate A and B are independent is symmetric.

Next we state four propositions:

(86)[12] Let E be a finite non empty set and A, B be events of E. If $0 < \mathrm{P}(B)$ and A and B are independent, then $\mathrm{P}(A/B) = \mathrm{P}(A)$.

(87) For every finite non empty set E and for all events A, B of E such that $\mathrm{P}(B) = 0$ holds A and B are independent.

(88) Let E be a finite non empty set and A, B be events of E. If A and B are independent, then A^{c} and B are independent.

(89) Let E be a finite non empty set and A, B be events of E. If A misses B and A and B are independent, then $\mathrm{P}(A) = 0$ or $\mathrm{P}(B) = 0$.

[12] The propositions (84) and (85) have been removed.

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