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On the Two Short Axiomatizations of Ortholattices

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Summary. In the paper, two short axiom systems for Boolean algebras are introduced. In the first section we show that the single axiom (DN_1) proposed in [2] in terms of disjunction and negation characterizes Boolean algebras. To prove that (DN_1) is a single axiom for Robbins algebras (that is, Boolean algebras as well), we use the Otter theorem prover. The second section contains proof that the two classical axioms ($\mathrm{Meredith}_1$), ($\mathrm{Meredith}_2$) proposed by Meredith [3] may also serve as a basis for Boolean algebras. The results will be used to characterize ortholattices.

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The articles [4] and [1] provide the notation and terminology for this paper.

1. Single Axiom for Boolean Algebras

Let L be a non empty complemented lattice structure. We say that L satisfies (DN_1) if and only if:

(Def. 1) For all elements x, y, z, u of L holds $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x+(z^{\mathrm{c}}+(z+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=z$.

Let us mention that TrivComplLat satisfies (DN_1) and TrivOrtLat satisfies (DN_1).

One can verify that there exists a non empty complemented lattice structure which is join-commutative and join-associative and satisfies (DN_1).

Next we state a number of propositions:

(1) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u, v be elements of L. Then $((x+y)^{\mathrm{c}}+(((z+u)^{\mathrm{c}}+x)^{\mathrm{c}}+(y^{\mathrm{c}}+(y+v)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=y$.

(2) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of L. Then $((x+y)^{\mathrm{c}}+((z+x)^{\mathrm{c}}+(y^{\mathrm{c}}+(y+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=y$.

(3) Let L be a non empty complemented lattice structure satisfying (DN_1) and x be an element of L. Then $((x+x^{\mathrm{c}})^{\mathrm{c}}+x)^{\mathrm{c}}=x^{\mathrm{c}}$.

(4) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of L. Then $((x+y)^{\mathrm{c}}+((z+x)^{\mathrm{c}}+(((y+y^{\mathrm{c}})^{\mathrm{c}}+y)^{\mathrm{c}}+(y+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=y$.

(5) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $((x+y)^{\mathrm{c}}+((z+x)^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}=y$.

(6) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}}+(x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}=y$.

(7) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(((x+y)^{\mathrm{c}}+x)^{\mathrm{c}}+(x+y)^{\mathrm{c}})^{\mathrm{c}}=x$.

(8) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+((x+y)^{\mathrm{c}}+x)^{\mathrm{c}})^{\mathrm{c}}=(x+y)^{\mathrm{c}}$.

(9) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x+z)^{\mathrm{c}})^{\mathrm{c}}=z$.

(10) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x+((y+z)^{\mathrm{c}}+(y+x)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=(y+x)^{\mathrm{c}}$.

(11) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $((((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}+y)^{\mathrm{c}}=(x^{\mathrm{c}}+y)^{\mathrm{c}}$.

(12) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x+((y+z)^{\mathrm{c}}+(z+x)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=(z+x)^{\mathrm{c}}$.

(13) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of L. Then $((x+y)^{\mathrm{c}}+((z+x)^{\mathrm{c}}+(y^{\mathrm{c}}+(u+y)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=y$.

(14) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y)^{\mathrm{c}}=(y+x)^{\mathrm{c}}$.

(15) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y)^{\mathrm{c}}+(y+z)^{\mathrm{c}})^{\mathrm{c}}+z)^{\mathrm{c}}=(y+z)^{\mathrm{c}}$.

(16) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $((x+((x+y)^{\mathrm{c}}+z)^{\mathrm{c}})^{\mathrm{c}}+z)^{\mathrm{c}}=((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}$.

(17) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(((x+y)^{\mathrm{c}}+x)^{\mathrm{c}}+y)^{\mathrm{c}}=(y+y)^{\mathrm{c}}$.

(18) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x^{\mathrm{c}}+(y+x)^{\mathrm{c}})^{\mathrm{c}}=x$.

(19) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}}+y^{\mathrm{c}})^{\mathrm{c}}=y$.

(20) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+(y+x^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=x^{\mathrm{c}}$.

(21) For every non empty complemented lattice structure L satisfying (DN_1) and for every element x of L holds $(x+x)^{\mathrm{c}}=x^{\mathrm{c}}$.

(22) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(((x+y)^{\mathrm{c}}+x)^{\mathrm{c}}+y)^{\mathrm{c}}=y^{\mathrm{c}}$.

(23) For every non empty complemented lattice structure L satisfying (DN_1) and for every element x of L holds $(x^{\mathrm{c}})^{\mathrm{c}}=x$.

(24) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}}+x)^{\mathrm{c}}+y=(y^{\mathrm{c}})^{\mathrm{c}}$.

(25) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}})^{\mathrm{c}}=y+x$.

(26) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+((y+z)^{\mathrm{c}}+(y+x)^{\mathrm{c}})^{\mathrm{c}}=((y+x)^{\mathrm{c}})^{\mathrm{c}}$.

(27) For every non empty complemented lattice structure L satisfying (DN_1) and for all elements x, y of L holds $x+y=y+x$.

Let us mention that every non empty complemented lattice structure which satisfies (DN_1) is also join-commutative.

The following propositions are true:

(28) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}}+x)^{\mathrm{c}}+y=y$.

(29) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $((x+y)^{\mathrm{c}}+y)^{\mathrm{c}}+x=x$.

(30) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $x+((y+x)^{\mathrm{c}}+y)^{\mathrm{c}}=x$.

(31) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y^{\mathrm{c}})^{\mathrm{c}}+(y^{\mathrm{c}}+y)^{\mathrm{c}}=(x+y^{\mathrm{c}})^{\mathrm{c}}$.

(32) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y)^{\mathrm{c}}+(y+y^{\mathrm{c}})^{\mathrm{c}}=(x+y)^{\mathrm{c}}$.

(33) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y)^{\mathrm{c}}+(y^{\mathrm{c}}+y)^{\mathrm{c}}=(x+y)^{\mathrm{c}}$.

(34) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(((x+y^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}+y)^{\mathrm{c}}=(y^{\mathrm{c}}+y)^{\mathrm{c}}$.

(35) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y^{\mathrm{c}}+y)^{\mathrm{c}}=(y^{\mathrm{c}}+y)^{\mathrm{c}}$.

(36) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y^{\mathrm{c}}+z)^{\mathrm{c}}+y)^{\mathrm{c}}+(y^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}=y$.

(37) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+((y+z)^{\mathrm{c}}+(y+x)^{\mathrm{c}})^{\mathrm{c}}=y+x$.

(38) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+(y+((z+y)^{\mathrm{c}}+x)^{\mathrm{c}})^{\mathrm{c}}=(z+y)^{\mathrm{c}}+x$.

(39) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+((y+x)^{\mathrm{c}}+(y+z)^{\mathrm{c}})^{\mathrm{c}}=y+x$.

(40) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $((x+y)^{\mathrm{c}}+((x+y)^{\mathrm{c}}+(x+z)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}+y=y$.

(41) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y^{\mathrm{c}}+z)^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}=y$.

(42) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+(y+x^{\mathrm{c}}+z)^{\mathrm{c}}=x$.

(43) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x^{\mathrm{c}}+(y+x+z)^{\mathrm{c}}=x^{\mathrm{c}}$.

(44) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+y)^{\mathrm{c}}+x=x+y^{\mathrm{c}}$.

(45) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of L. Then $(x+(x+y^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=(x+y)^{\mathrm{c}}$.

(46) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $((x+y)^{\mathrm{c}}+(x+z))^{\mathrm{c}}+y=y$.

(47) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}+y=((x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}$.

(48) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}+y=x^{\mathrm{c}}+y$.

(49) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x^{\mathrm{c}}+(((y+x)^{\mathrm{c}})^{\mathrm{c}}+(y+z))^{\mathrm{c}})^{\mathrm{c}}+(y+z)=((y+x)^{\mathrm{c}})^{\mathrm{c}}+(y+z)$.

(50) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x^{\mathrm{c}}+(y+x+(y+z))^{\mathrm{c}})^{\mathrm{c}}+(y+z)=((y+x)^{\mathrm{c}})^{\mathrm{c}}+(y+z)$.

(51) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x^{\mathrm{c}}+(y+x+(y+z))^{\mathrm{c}})^{\mathrm{c}}+(y+z)=(y+x)+(y+z)$.

(52) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x^{\mathrm{c}})^{\mathrm{c}}+(y+z)=(y+x)+(y+z)$.

(53) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x+y)+(x+z)=y+(x+z)$.

(54) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x+y)+(x+z)=z+(x+y)$.

(55) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+(y+z)=z+(y+x)$.

(56) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $x+(y+z)=y+(z+x)$.

(57) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of L. Then $(x+y)+z=x+(y+z)$.

One can check that every non empty complemented lattice structure which satisfies (DN_1) is also join-associative and every non empty complemented lattice structure which satisfies (DN_1) is also Robbins.

One can prove the following two propositions:

(58) Let L be a non empty complemented lattice structure and x, z be elements of L. Suppose L is join-commutative, join-associative, and Huntington. Then $(z+x)*(z+x^{\mathrm{c}})=z$.

(59) Let L be a non empty complemented lattice structure such that L is join-commutative, join-associative, and Robbins. Then L satisfies (DN_1).

One can verify that every non empty complemented lattice structure which is join-commutative, join-associative, and Robbins satisfies also (DN_1).

One can verify that there exists a pre-ortholattice which is de Morgan and satisfies (DN_1).

One can check that every pre-ortholattice which is de Morgan satisfies (DN_1) is also Boolean and every well-complemented pre-ortholattice which is Boolean satisfies also (DN_1).

2. MEREDITH TWO AXIOMS FOR BOOLEAN ALGEBRAS

Let L be a non empty complemented lattice structure. We say that L satisfies $(\mathrm{Meredith}_1)$ if and only if:

(Def. 2) For all elements x, y of L holds $(x^{\mathrm{c}}+y)^{\mathrm{c}}+x=x$.

We say that L satisfies $(\mathrm{Meredith}_2)$ if and only if:

(Def. 3) For all elements x, y, z of L holds $(x^{\mathrm{c}}+y)^{\mathrm{c}}+(z+y)=y+(z+x)$.

Let us observe that every non empty complemented lattice structure which satisfies $(\mathrm{Meredith}_1)$ and $(\mathrm{Meredith}_2)$ is also join-commutative, join-associative, and Huntington and every non empty complemented lattice structure which is join-commutative, join-associative, and Huntington satisfies also $(\mathrm{Meredith}_1)$ and $(\mathrm{Meredith}_2)$.

Let us observe that there exists a pre-ortholattice which is de Morgan and satisfies $(\mathrm{Meredith}_1)$, $(\mathrm{Meredith}_2)$, and (DN_1).

One can check that every pre-ortholattice which is de Morgan satisfies $(\mathrm{Meredith}_1)$ and $(\mathrm{Meredith}_2)$ is also Boolean and every well-complemented pre-ortholattice which is Boolean satisfies also $(\mathrm{Meredith}_1)$ and $(\mathrm{Meredith}_2)$.

References

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