JOURNAL OF FORMALIZED MATHEMATICS
Volume 13, Released 2001, Published 2003
Inst. of Computer Science, Univ. of Białystok

Robbins Algebras vs. Boolean Algebras[1]

Adam Grabowski
University of Białystok

Summary. In the early 1930s, Huntington proposed several axiom systems for Boolean algebras. Robbins slightly changed one of them and asked if the resulted system is still a basis for variety of Boolean algebras. The solution (afirmative answer) was given in 1996 by McCune with the help of automated theorem prover EQP/OTTER. Some simplified and restucturized versions of this proof are known. In our version of proof that all Robbins algebras are Boolean we use the results of McCune [8], Huntington [5], [7], [6] and Dahn [4].

MML Identifier: ROBBINS1.

WWW: http://mizar.org/JFM/Vol13/robbins1.html

The articles [11], [12], [10], [1], [2], [3], and [9] provide the notation and terminology for this paper.

1. PRELIMINARIES

We introduce ComplStr which are extensions of 1-sorted structure and are systems
⟨ a carrier, a complement operation ⟩,
where the carrier is a set and the complement operation is a unary operation on the carrier.

We consider complemented lattice structures as extensions of ⊔-semi lattice structure and ComplStr as systems
⟨ a carrier, a join operation, a complement operation ⟩,
where the carrier is a set, the join operation is a binary operation on the carrier, and the complement operation is a unary operation on the carrier.

We introduce ortholattice structures which are extensions of complemented lattice structure and lattice structure and are systems
⟨ a carrier, a join operation, a meet operation, a complement operation ⟩,
where the carrier is a set, the join operation and the meet operation are binary operations on the carrier, and the complement operation is a unary operation on the carrier.

The strict complemented lattice structure TrivComplLat is defined as follows:

(Def. 1) $\mathrm{TrivComplLat} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_1 \rangle$.

The strict ortholattice structure TrivOrtLat is defined as follows:

(Def. 2) $\mathrm{TrivOrtLat} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_2, \mathrm{op}_1 \rangle$.

Let us note that TrivComplLat is non empty and trivial and TrivOrtLat is non empty and trivial.

Let us observe that there exists an ortholattice structure which is strict, non empty, and trivial and there exists a complemented lattice structure which is strict, non empty, and trivial.

Let L be a non empty trivial complemented lattice structure. One can check that the ComplStr of L is non empty and trivial.

[1]This work has been partially supported by TYPES grant IST-1999-29001.

Let us note that there exists a ComplStr which is strict, non empty, and trivial.

Let L be a non empty ComplStr and let x be an element of L. The functor $x^{\rm c}$ yields an element of L and is defined as follows:

(Def. 3) $x^{\rm c} =$ (the complement operation of $L)(x)$.

Let L be a non empty complemented lattice structure and let x, y be elements of L. We introduce $x+y$ as a synonym of $x \sqcup y$.

Let L be a non empty complemented lattice structure and let x, y be elements of L. The functor $x*y$ yielding an element of L is defined by:

(Def. 4) $x*y = (x^{\rm c} \sqcup y^{\rm c})^{\rm c}$.

Let L be a non empty complemented lattice structure. We say that L is Robbins if and only if:

(Def. 5) For all elements x, y of L holds $((x+y)^{\rm c} + (x+y^{\rm c})^{\rm c})^{\rm c} = x$.

We say that L is Huntington if and only if:

(Def. 6) For all elements x, y of L holds $(x^{\rm c}+y^{\rm c})^{\rm c} + (x^{\rm c}+y)^{\rm c} = x$.

Let G be a non empty $\sqcup$-semi lattice structure. We say that G is join-idempotent if and only if:

(Def. 7) For every element x of G holds $x \sqcup x = x$.

Let us mention that TrivComplLat is join-commutative, join-associative, Robbins, Huntington, and join-idempotent and TrivOrtLat is join-commutative, join-associative, Huntington, and Robbins.

Let us observe that TrivOrtLat is meet-commutative, meet-associative, meet-absorbing, and join-absorbing.

Let us mention that there exists a non empty complemented lattice structure which is strict, join-associative, join-commutative, Robbins, join-idempotent, and Huntington.

One can verify that there exists a non empty ortholattice structure which is strict, lattice-like, Robbins, and Huntington.

Let L be a join-commutative non empty complemented lattice structure and let x, y be elements of L. Let us observe that the functor $x+y$ is commutative.

One can prove the following propositions:

(1) Let L be a Huntington join-commutative join-associative non empty complemented lattice structure and a, b be elements of L. Then $a*b + a*b^{\rm c} = a$.

(2) Let L be a Huntington join-commutative join-associative non empty complemented lattice structure and a be an element of L. Then $a + a^{\rm c} = a^{\rm c} + (a^{\rm c})^{\rm c}$.

(3) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and x be an element of L. Then $(x^{\rm c})^{\rm c} = x$.

(4) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. Then $a + a^{\rm c} = b + b^{\rm c}$.

(5) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then there exists an element c of L such that for every element a of L holds $c + a = c$ and $a + a^{\rm c} = c$.

(6) Every join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure is upper-bounded.

Let us note that every non empty complemented lattice structure which is join-commutative, join-associative, join-idempotent, and Huntington is also upper-bounded.

Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then $\top_L$ can be characterized by the condition:

(Def. 8) There exists an element a of L such that $\top_L = a + a^{\rm c}$.

We now state two propositions:

(7) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then there exists an element c of L such that for every element a of L holds $c * a = c$ and $(a + a^{\rm c})^{\rm c} = c$.

(8) Let L be a join-commutative join-associative non empty complemented lattice structure and a, b be elements of L. Then $a * b = b * a$.

Let L be a join-commutative join-associative non empty complemented lattice structure and let x, y be elements of L. Let us observe that the functor $x * y$ is commutative.

Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. The functor $\bot_L^{\rm C}$ yields an element of L and is defined by:

(Def. 9) For every element a of L holds $\bot_L^{\rm C} * a = \bot_L^{\rm C}$.

Next we state several propositions:

(9) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure and a be an element of L. Then $\bot_L^{\rm C} = (a + a^{\rm c})^{\rm c}$.

(10) Let L be a join-commutative join-associative join-idempotent Huntington non empty complemented lattice structure. Then $(\top_L)^{\rm c} = \bot_L^{\rm C}$ and $\top_L = (\bot_L^{\rm C})^{\rm c}$.

(11) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. If $a^{\rm c} = b^{\rm c}$, then $a = b$.

(12) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. Then $a + (b + b^{\rm c})^{\rm c} = a$.

(13) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of L. Then $a + a = a$.

Let us observe that every non empty complemented lattice structure which is join-commutative, join-associative, and Huntington is also join-idempotent.

One can prove the following propositions:

(14) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of L. Then $a + \bot_L^{\rm C} = a$.

(15) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of L. Then $a * \top_L = a$.

(16) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of L. Then $a * a^{\rm c} = \bot_L^{\rm C}$.

(17) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of L. Then $a * (b * c) = (a * b) * c$.

(18) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. Then $a + b = (a^{\rm c} * b^{\rm c})^{\rm c}$.

(19) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of L. Then $a * a = a$.

(20) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a be an element of L. Then $a + \top_L = \top_L$.

(21) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. Then $a + a * b = a$.

(22) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. Then $a * (a + b) = a$.

(23) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. If $a^{\mathrm{c}} + b = \top_L$ and $b^{\mathrm{c}} + a = \top_L$, then $a = b$.

(24) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. If $a + b = \top_L$ and $a * b = \perp_L^{\mathrm{C}}$, then $a^{\mathrm{c}} = b$.

(25) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of L. Then $a * b * c + a * b * c^{\mathrm{c}} + a * b^{\mathrm{c}} * c + a * b^{\mathrm{c}} * c^{\mathrm{c}} + a^{\mathrm{c}} * b * c + a^{\mathrm{c}} * b * c^{\mathrm{c}} + a^{\mathrm{c}} * b^{\mathrm{c}} * c + a^{\mathrm{c}} * b^{\mathrm{c}} * c^{\mathrm{c}} = \top_L$.

(26) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of L. Then

(i) $a * c * (b * c^{\mathrm{c}}) = \perp_L^{\mathrm{C}}$,

(ii) $a * b * c * (a^{\mathrm{c}} * b * c) = \perp_L^{\mathrm{C}}$,

(iii) $a * b^{\mathrm{c}} * c * (a^{\mathrm{c}} * b * c) = \perp_L^{\mathrm{C}}$,

(iv) $a * b * c * (a^{\mathrm{c}} * b^{\mathrm{c}} * c) = \perp_L^{\mathrm{C}}$,

(v) $a * b * c^{\mathrm{c}} * (a^{\mathrm{c}} * b^{\mathrm{c}} * c^{\mathrm{c}}) = \perp_L^{\mathrm{C}}$, and

(vi) $a * b^{\mathrm{c}} * c * (a^{\mathrm{c}} * b * c) = \perp_L^{\mathrm{C}}$.

(27) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of L. Then $a * b + a * c = a * b * c + a * b * c^{\mathrm{c}} + a * b^{\mathrm{c}} * c$.

(28) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of L. Then $(a * (b + c))^{\mathrm{c}} = a * b^{\mathrm{c}} * c^{\mathrm{c}} + a^{\mathrm{c}} * b * c + a^{\mathrm{c}} * b * c^{\mathrm{c}} + a^{\mathrm{c}} * b^{\mathrm{c}} * c + a^{\mathrm{c}} * b^{\mathrm{c}} * c^{\mathrm{c}}$.

(29) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of L. Then $a * b + a * c + (a * (b + c))^{\mathrm{c}} = \top_L$.

(30) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of L. Then $(a * b + a * c) * (a * (b + c))^{\mathrm{c}} = \perp_L^{\mathrm{C}}$.

(31) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of L. Then $a * (b + c) = a * b + a * c$.

(32) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b, c be elements of L. Then $a + b * c = (a + b) * (a + c)$.

2. PRE-ORTHOLATTICES

Let L be a non empty ortholattice structure. We say that L is well-complemented if and only if:

(Def. 10) For every element a of L holds a^{c} is a complement of a.

Let us note that TrivOrtLat is Boolean and well-complemented.

A pre-ortholattice is a lattice-like non empty ortholattice structure.

Let us note that there exists a pre-ortholattice which is strict, Boolean, and well-complemented.

One can prove the following two propositions:

(33) For every distributive well-complemented pre-ortholattice L and for every element x of L holds $(x^{\mathrm{c}})^{\mathrm{c}} = x$.

(34) Let L be a bounded distributive well-complemented pre-ortholattice and x, y be elements of L. Then $x \sqcap y = (x^{\mathrm{c}} \sqcup y^{\mathrm{c}})^{\mathrm{c}}$.

3. Correspondence between Boolean Pre-OrthoLattices and Boolean Lattices

Let L be a non empty complemented lattice structure. The functor CLatt L yielding a strict ortholattice structure is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of CLatt $L =$ the carrier of L,

(ii) the join operation of CLatt $L =$ the join operation of L,

(iii) the complement operation of CLatt $L =$ the complement operation of L, and

(iv) for all elements a, b of L holds (the meet operation of CLatt $L)(a, b) = a * b$.

Let L be a non empty complemented lattice structure. Note that CLatt L is non empty.

Let L be a join-commutative non empty complemented lattice structure. One can verify that CLatt L is join-commutative.

Let L be a join-associative non empty complemented lattice structure. One can verify that CLatt L is join-associative.

Let L be a join-commutative join-associative non empty complemented lattice structure. Observe that CLatt L is meet-commutative.

Next we state the proposition

(35) Let L be a non empty complemented lattice structure, a, b be elements of L, and a', b' be elements of CLatt L. If $a = a'$ and $b = b'$, then $a * b = a' \sqcap b'$ and $a + b = a' \sqcup b'$ and $a^{\mathrm{c}} = a'^{\mathrm{c}}$.

Let L be a join-commutative join-associative Huntington non empty complemented lattice structure. Observe that CLatt L is meet-associative, join-absorbing, and meet-absorbing.

Let L be a Huntington non empty complemented lattice structure. One can verify that CLatt L is Huntington.

Let L be a join-commutative join-associative Huntington non empty complemented lattice structure. Note that CLatt L is lower-bounded.

The following proposition is true

(36) For every join-commutative join-associative Huntington non empty complemented lattice structure L holds $\perp_L^{\mathrm{C}} = \perp_{\mathrm{CLatt}\,L}$.

Let L be a join-commutative join-associative Huntington non empty complemented lattice structure. One can verify that CLatt L is complemented, distributive, and bounded.

4. Proofs according to Bernd Ingo Dahn

Let G be a non empty complemented lattice structure and let x be an element of G. We introduce $-x$ as a synonym of x^{c}.

Let G be a join-commutative non empty complemented lattice structure. Let us observe that G is Huntington if and only if:

(Def. 12) For all elements x, y of G holds $-(-x + -y) + -(x + -y) = y$.

Let G be a non empty complemented lattice structure. We say that G has idempotent element if and only if:

(Def. 13) There exists an element x of G such that $x + x = x$.

In the sequel G is a Robbins join-associative join-commutative non empty complemented lattice structure and x, y, z are elements of G.

Let G be a non empty complemented lattice structure and let x, y be elements of G. The functor $\delta(x,y)$ yielding an element of G is defined as follows:

(Def. 14) $\delta(x,y) = -(-x + y)$.

Let G be a non empty complemented lattice structure and let x, y be elements of G. The functor $\mathrm{Expand}(x,y)$ yields an element of G and is defined as follows:

(Def. 15) $\mathrm{Expand}(x,y) = \delta(x+y,\delta(x,y))$.

Let G be a non empty complemented lattice structure and let x be an element of G. The functor x_0 yields an element of G and is defined as follows:

(Def. 16) $x_0 = -(-x+x)$.

The functor $2x$ yields an element of G and is defined as follows:

(Def. 17) $2x = x+x$.

Let G be a non empty complemented lattice structure and let x be an element of G. The functor x_1 yields an element of G and is defined by:

(Def. 18) $x_1 = x_0 + x$.

The functor x_2 yielding an element of G is defined as follows:

(Def. 19) $x_2 = x_0 + 2x$.

The functor x_3 yielding an element of G is defined as follows:

(Def. 20) $x_3 = x_0 + (2x + x)$.

The functor x_4 yielding an element of G is defined by:

(Def. 21) $x_4 = x_0 + (2x + 2x)$.

Next we state a number of propositions:

(37) $\delta(x+y,\delta(x,y)) = y$.

(38) $\mathrm{Expand}(x,y) = y$.

(39) $\delta(-x+y,z) = -(\delta(x,y)+z)$.

(40) $\delta(x,x) = x_0$.

(41) $\delta(2x,x_0) = x$.

(42) $\delta(x_2,x) = x_0$.

(43) $x_2 + x = x_3$.

(44) $x_4 + x_0 = x_3 + x_1$.

(45) $x_3 + x_0 = x_2 + x_1$.

(46) $x_3 + x = x_4$.

(47) $\delta(x_3,x_0) = x$.

(48) If $-x = -y$, then $\delta(x,z) = \delta(y,z)$.

(49) $\delta(x,-y) = \delta(y,-x)$.

(50) $\delta(x_3,x) = x_0$.

(51) $\delta(x_1+x_3,x) = x_0$.

(52) $\delta(x_1+x_2,x) = x_0$.

(53) $\delta(x_1+x_3,x_0) = x$.

Let us consider G, x. The functor $\beta(x)$ yields an element of G and is defined by:

(Def. 22) $\beta(x) = -(x_1 + x_3) + x + -x_3$.

We now state three propositions:

(54) $\delta(\beta(x), x) = -x_3$.

(55) $\delta(\beta(x), x) = -(x_1 + x_3)$.

(56) There exist y, z such that $-(y + z) = -z$.

5. Proofs according to William McCune

We now state two propositions:

(57) If for every z holds $--z = z$, then G is Huntington.

(58) If G has idempotent element, then G is Huntington.

Let us note that TrivComplLat has idempotent element.

One can check that every Robbins join-associative join-commutative non empty complemented lattice structure which has idempotent element is also Huntington.

We now state two propositions:

(59) If there exist elements c, d of G such that $c + d = c$, then G is Huntington.

(60) There exist y, z such that $y + z = z$.

Let us observe that every join-associative join-commutative non empty complemented lattice structure which is Robbins is also Huntington.

Let L be a non empty ortholattice structure. We say that L is de Morgan if and only if:

(Def. 23) For all elements x, y of L holds $x \sqcap y = (x^{c} \sqcup y^{c})^{c}$.

Let L be a non empty complemented lattice structure. One can verify that CLatt L is de Morgan.

Next we state two propositions:

(61) Let L be a well-complemented join-commutative meet-commutative non empty ortholattice structure and x be an element of L. Then $x + x^{c} = \top_L$ and $x \sqcap x^{c} = \bot_L$.

(62) For every bounded distributive well-complemented pre-ortholattice L holds $(\top_L)^{c} = \bot_L$.

Let us note that TrivOrtLat is de Morgan.

Let us note that there exists a pre-ortholattice which is strict, de Morgan, Boolean, Robbins, and Huntington.

One can verify that every non empty ortholattice structure which is join-associative, join-commutative, and de Morgan is also meet-commutative.

The following proposition is true

(63) For every Huntington de Morgan pre-ortholattice L holds $\bot_L^{\mathrm{C}} = \bot_L$.

Let us mention that every well-complemented pre-ortholattice which is Boolean is also Huntington.

Let us observe that every de Morgan pre-ortholattice which is Huntington is also Boolean.

Let us note that every pre-ortholattice which is Robbins and de Morgan is also Boolean and every well-complemented pre-ortholattice which is Boolean is also Robbins.

REFERENCES

[1] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[4] B. I. Dahn. Robbins algebras are Boolean: A revision of McCune's computer-generated solution of Robbins problem. *Journal of Algebra*, 208:526–532, 1998.

[5] E. V. Huntington. Sets of independent postulates for the algebra of logic. *Trans. AMS*, 5:288–309, 1904.

[6] E. V. Huntington. Boolean algebra. A correction. *Trans. AMS*, 35:557–558, 1933.

[7] E. V. Huntington. New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's Principia Mathematica. *Trans. AMS*, 35:274–304, 1933.

[8] W. McCune. Solution of the Robbins problem. *Journal of Automated Reasoning*, 19:263–276, 1997.

[9] Michał Muzalewski. Midpoint algebras. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/midsp_1.html.

[10] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_2.html.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[12] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/lattices.html.

Received June 12, 2001

Published January 2, 2004