JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Submodules and Cosets of Submodules in Right Module over Associative Ring

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MML Identifier: RMOD_2.

WWW: http://mizar.org/JFM/Vol2/rmod_2.html

The articles [7], [3], [9], [10], [1], [2], [6], [8], [4], and [5] provide the notation and terminology for this paper.

For simplicity, we follow the rules: x is a set, R is a ring, a is a scalar of R, V, X, Y are right modules over R, u, v, v_1, v_2 are vectors of V, and V_1, V_2, V_3 are subsets of V.

Let us consider R, V, V_1. We say that V_1 is linearly closed if and only if:

(Def. 1) For all v, u such that $v \in V_1$ and $u \in V_1$ holds $v+u \in V_1$ and for all a, v such that $v \in V_1$ holds $v \cdot a \in V_1$.

We now state several propositions:

(4)[1] If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $0_V \in V_1$.

(5) If V_1 is linearly closed, then for every v such that $v \in V_1$ holds $-v \in V_1$.

(6) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v-u \in V_1$.

(7) $\{0_V\}$ is linearly closed.

(8) If the carrier of $V = V_1$, then V_1 is linearly closed.

(9) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v+u : v \in V_1 \land u \in V_2\}$, then V_3 is linearly closed.

(10) If V_1 is linearly closed and V_2 is linearly closed, then $V_1 \cap V_2$ is linearly closed.

Let us consider R and let us consider V. A right module over R is said to be a submodule of V if it satisfies the conditions (Def. 2).

(Def. 2)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the zero of it $=$ the zero of V,

(iii) the addition of it $=$ (the addition of V) $\restriction$ [: the carrier of it, the carrier of it :], and

(iv) the right multiplication of it $=$ (the right multiplication of V) $\restriction$ [: the carrier of it, the carrier of R :].

[1] The propositions (1)–(3) have been removed.

We follow the rules: W, W_1, W_2 denote submodules of V and w, w_1, w_2 denote vectors of W.
One can prove the following propositions:

(16)[2] If $x \in W_1$ and W_1 is a submodule of W_2, then $x \in W_2$.

(17) If $x \in W$, then $x \in V$.

(18) w is a vector of V.

(19) $0_W = 0_V$.

(20) $0_{(W_1)} = 0_{(W_2)}$.

(21) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(22) If $w = v$, then $w \cdot a = v \cdot a$.

(23) If $w = v$, then $-v = -w$.

(24) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(25) $0_V \in W$.

(26) $0_{(W_1)} \in W_2$.

(27) $0_W \in V$.

(28) If $u \in W$ and $v \in W$, then $u + v \in W$.

(29) If $v \in W$, then $v \cdot a \in W$.

(30) If $v \in W$, then $-v \in W$.

(31) If $u \in W$ and $v \in W$, then $u - v \in W$.

(32) V is a submodule of V.

(33) Let X, V be strict right modules over R. If V is a submodule of X and X is a submodule of V, then $V = X$.

Let us consider R, V. One can verify that there exists a submodule of V which is strict.
Next we state several propositions:

(34) If V is a submodule of X and X is a submodule of Y, then V is a submodule of Y.

(35) If the carrier of $W_1 \subseteq$ the carrier of W_2, then W_1 is a submodule of W_2.

(36) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a submodule of W_2.

(37) For all strict submodules W_1, W_2 of V such that the carrier of $W_1 =$ the carrier of W_2 holds $W_1 = W_2$.

(38) For all strict submodules W_1, W_2 of V such that for every vector v of V holds $v \in W_1$ iff $v \in W_2$ holds $W_1 = W_2$.

(39) Let V be a strict right module over R and W be a strict submodule of V. If the carrier of $W =$ the carrier of V, then $W = V$.

(40) Let V be a strict right module over R and W be a strict submodule of V. If for every vector v of V holds $v \in W$, then $W = V$.

(41) If the carrier of $W = V_1$, then V_1 is linearly closed.

[2] The propositions (11)–(15) have been removed.

(42) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists a strict submodule W of V such that $V_1 =$ the carrier of W.

Let us consider R and let us consider V. The functor $\mathbf{0}_V$ yields a strict submodule of V and is defined as follows:

(Def. 3) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let us consider R and let us consider V. The functor Ω_V yields a strict submodule of V and is defined as follows:

(Def. 4) $\Omega_V =$ the right module structure of V.

One can prove the following propositions:

(46)[3] $x \in \mathbf{0}_V$ iff $x = 0_V$.

(47) $\mathbf{0}_W = \mathbf{0}_V$.

(48) $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(49) $\mathbf{0}_W$ is a submodule of V.

(50) $\mathbf{0}_V$ is a submodule of W.

(51) $\mathbf{0}_{(W_1)}$ is a submodule of W_2.

(53)[4] Every strict right module V over R is a submodule of Ω_V.

Let us consider R, let us consider V, and let us consider v, W. The functor $v+W$ yields a subset of V and is defined by:

(Def. 5) $v+W = \{v+u : u \in W\}$.

Let us consider R, let us consider V, and let us consider W. A subset of V is called a coset of W if:

(Def. 6) There exists v such that it $= v+W$.

In the sequel B, C denote cosets of W.

One can prove the following propositions:

(57)[5] $x \in v+W$ iff there exists u such that $u \in W$ and $x = v+u$.

(58) $0_V \in v+W$ iff $v \in W$.

(59) $v \in v+W$.

(60) $0_V + W =$ the carrier of W.

(61) $v + \mathbf{0}_V = \{v\}$.

(62) $v + \Omega_V =$ the carrier of V.

(63) $0_V \in v+W$ iff $v+W =$ the carrier of W.

(64) $v \in W$ iff $v+W =$ the carrier of W.

(65) If $v \in W$, then $v \cdot a + W =$ the carrier of W.

(66) $u \in W$ iff $v+W = v+u+W$.

[3] The propositions (43)–(45) have been removed.
[4] The proposition (52) has been removed.
[5] The propositions (54)–(56) have been removed.

(67) $u \in W$ iff $v+W = (v-u)+W$.

(68) $v \in u+W$ iff $u+W = v+W$.

(69) If $u \in v_1+W$ and $u \in v_2+W$, then $v_1+W = v_2+W$.

(70) If $v \in W$, then $v \cdot a \in v+W$.

(71) If $v \in W$, then $-v \in v+W$.

(72) $u+v \in v+W$ iff $u \in W$.

(73) $v-u \in v+W$ iff $u \in W$.

(75)[6] $u \in v+W$ iff there exists v_1 such that $v_1 \in W$ and $u = v-v_1$.

(76) There exists v such that $v_1 \in v+W$ and $v_2 \in v+W$ iff $v_1 - v_2 \in W$.

(77) If $v+W = u+W$, then there exists v_1 such that $v_1 \in W$ and $v+v_1 = u$.

(78) If $v+W = u+W$, then there exists v_1 such that $v_1 \in W$ and $v-v_1 = u$.

(79) For all strict submodules W_1, W_2 of V holds $v+W_1 = v+W_2$ iff $W_1 = W_2$.

(80) For all strict submodules W_1, W_2 of V such that $v+W_1 = u+W_2$ holds $W_1 = W_2$.

(81) There exists C such that $v \in C$.

(82) C is linearly closed iff $C =$ the carrier of W.

(83) For all strict submodules W_1, W_2 of V and for every coset C_1 of W_1 and for every coset C_2 of W_2 such that $C_1 = C_2$ holds $W_1 = W_2$.

(84) $\{v\}$ is a coset of $\mathbf{0}_V$.

(85) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.

(86) The carrier of W is a coset of W.

(87) The carrier of V is a coset of Ω_V.

(88) If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of V.

(89) $0_V \in C$ iff $C =$ the carrier of W.

(90) $u \in C$ iff $C = u+W$.

(91) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u+v_1 = v$.

(92) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u-v_1 = v$.

(93) There exists C such that $v_1 \in C$ and $v_2 \in C$ iff $v_1 - v_2 \in W$.

(94) If $u \in B$ and $u \in C$, then $B = C$.

[6] The proposition (74) has been removed.

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Received October 22, 1990

Published January 2, 2004