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Basis of Real Linear Space

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Summary. Notions of linear independence and dependence of set of vectors, the subspace generated by a set of vectors and basis of real linear space are introduced. Some theorems concerning those notions are proved.

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The articles [8], [6], [16], [10], [3], [17], [2], [4], [5], [14], [9], [7], [13], [12], [11], [15], and [1] provide the notation and terminology for this paper.

For simplicity, we follow the rules: x, X, Y, Z are sets, a, b are real numbers, V is a real linear space, W_1, W_2, W_3 are subspaces of V, v, v_1, v_2 are vectors of V, A, B are subsets of V, L, L_1, L_2 are linear combinations of V, l is a linear combination of A, F, G are finite sequences of elements of the carrier of V, f is a function from the carrier of V into $\mathbb{R}$, M is a non empty set, and C_1 is a choice function of M.

The following four propositions are true:

(1) $\sum(L_1+L_2) = \sum L_1 + \sum L_2$.

(2) $\sum(a \cdot L) = a \cdot \sum L$.

(3) $\sum(-L) = -\sum L$.

(4) $\sum(L_1-L_2) = \sum L_1 - \sum L_2$.

Let us consider V and let us consider A. We say that A is linearly independent if and only if:

(Def. 1) For every l such that $\sum l = 0_V$ holds the support of $l = \emptyset$.

We introduce A is linearly dependent as an antonym of A is linearly independent.

Next we state several propositions:

(6)[1] If $A \subseteq B$ and B is linearly independent, then A is linearly independent.

(7) If A is linearly independent, then $0_V \notin A$.

(8) $\emptyset_{\text{the carrier of } V}$ is linearly independent.

(9) $\{v\}$ is linearly independent iff $v \neq 0_V$.

(10) $\{0_V\}$ is linearly dependent.

(11) If $\{v_1, v_2\}$ is linearly independent, then $v_1 \neq 0_V$ and $v_2 \neq 0_V$.

[1] The proposition (5) has been removed.

(12) $\{v, 0_V\}$ is linearly dependent and $\{0_V, v\}$ is linearly dependent.

(13) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent iff $v_2 \neq 0_V$ and for every a holds $v_1 \neq a \cdot v_2$.

(14) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent iff for all a, b such that $a \cdot v_1 + b \cdot v_2 = 0_V$ holds $a = 0$ and $b = 0$.

Let us consider V and let us consider A. The functor $\mathrm{Lin}(A)$ yields a strict subspace of V and is defined by:

(Def. 2) The carrier of $\mathrm{Lin}(A) = \{\sum l\}$.

Next we state a number of propositions:

(17)[2] $x \in \mathrm{Lin}(A)$ iff there exists l such that $x = \sum l$.

(18) If $x \in A$, then $x \in \mathrm{Lin}(A)$.

(19) $\mathrm{Lin}(\emptyset_{\text{the carrier of } V}) = \mathbf{0}_V$.

(20) If $\mathrm{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{0_V\}$.

(21) For every strict subspace W of V such that $A =$ the carrier of W holds $\mathrm{Lin}(A) = W$.

(22) For every strict real linear space V and for every subset A of V such that $A =$ the carrier of V holds $\mathrm{Lin}(A) = V$.

(23) If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(24) For every strict real linear space V and for all subsets A, B of V such that $\mathrm{Lin}(A) = V$ and $A \subseteq B$ holds $\mathrm{Lin}(B) = V$.

(25) $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(26) $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(27) Suppose A is linearly independent. Then there exists B such that $A \subseteq B$ and B is linearly independent and $\mathrm{Lin}(B) =$ the RLS structure of V.

(28) If $\mathrm{Lin}(A) = V$, then there exists B such that $B \subseteq A$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

Let V be a real linear space. A subset of V is called a basis of V if:

(Def. 3) It is linearly independent and $\mathrm{Lin}(\mathrm{it}) =$ the RLS structure of V.

In the sequel I denotes a basis of V.
The following propositions are true:

(32)[3] Let V be a strict real linear space and A be a subset of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

(33) If $\mathrm{Lin}(A) = V$, then there exists I such that $I \subseteq A$.

(34) If $Z \neq \emptyset$ and Z is finite and for all X, Y such that $X \in Z$ and $Y \in Z$ holds $X \subseteq Y$ or $Y \subseteq X$, then $\bigcup Z \in Z$.

(35) If $\emptyset \notin M$, then $\mathrm{dom}\, C_1 = M$ and $\mathrm{rng}\, C_1 \subseteq \bigcup M$.

(36) $x \in \mathbf{0}_V$ iff $x = 0_V$.

(37) If W_1 is a subspace of W_3, then $W_1 \cap W_2$ is a subspace of W_3.

[2] The propositions (15) and (16) have been removed.
[3] The propositions (29)–(31) have been removed.

(38) If W_1 is a subspace of W_2 and a subspace of W_3, then W_1 is a subspace of $W_2 \cap W_3$.

(39) If W_1 is a subspace of W_3 and W_2 is a subspace of W_3, then $W_1 + W_2$ is a subspace of W_3.

(40) If W_1 is a subspace of W_2, then W_1 is a subspace of $W_2 + W_3$.

(41) $f\,(F \frown G) = (f\,F) \frown (f\,G)$.

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