JOURNAL OF FORMALIZED MATHEMATICS
Volume 12, Released 2000, Published 2003
Inst. of Computer Science, Univ. of Białystok

Introduction to Several Concepts of Convexity and Semicontinuity for Function from $\mathbb{R}$ to $\mathbb{R}$

Noboru Endou
Shinshu University
Nagano

Katsumi Wasaki
Shinshu University
Nagano

Yasunari Shidama
Shinshu University
Nagano

Summary. This article is an introduction to convex analysis. In the beginning, we have defined the concept of strictly convexity and proved some basic properties between convexity and strictly convexity. Moreover, we have defined concepts of other convexity and semicontinuity, and proved their basic properties.

MML Identifier: RFUNCT_4.

WWW: http://mizar.org/JFM/Vol12/rfunct_4.html

The articles [16], [18], [1], [19], [5], [2], [10], [13], [8], [17], [6], [7], [11], [9], [15], [12], [3], [4], and [14] provide the notation and terminology for this paper.

1. Some Useful Properties of n-Tuples on $\mathbb{R}$

We adopt the following rules: a, b, r, s, x_0, x are real numbers, f, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and X, Y are sets.

We now state several propositions:

(1) For all real numbers a, b holds $\max(a,b) \geq \min(a,b)$.

(2) Let n be a natural number, R_1, R_2 be elements of $\mathbb{R}^n$, and r_1, r_2 be real numbers. Then $R_1 \frown \langle r_1 \rangle \bullet R_2 \frown \langle r_2 \rangle = (R_1 \bullet R_2) \frown \langle r_1 \cdot r_2 \rangle$.

(3) Let n be a natural number and R be an element of $\mathbb{R}^n$. Suppose $\sum R = 0$ and for every natural number i such that $i \in \operatorname{dom} R$ holds $0 \leq R(i)$. Let i be a natural number. If $i \in \operatorname{dom} R$, then $R(i) = 0$.

(4) Let n be a natural number and R be an element of $\mathbb{R}^n$. Suppose that for every natural number i such that $i \in \operatorname{dom} R$ holds $0 = R(i)$. Then $R = n \mapsto (0$ **qua** real number$)$.

(5) For every natural number n and for every element R of $\mathbb{R}^n$ holds $n \mapsto (0$ **qua** real number$)$ $\bullet R = n \mapsto (0$ **qua** real number$)$.

2. Convex and Strictly Convex Functions

Let us consider f, X. We say that f is strictly convex on X if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every real number p such that $0 < p$ and $p < 1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r + (1-p) \cdot s \in X$ and $r \neq s$ holds $f(p \cdot r + (1-p) \cdot s) < p \cdot f(r) + (1-p) \cdot f(s)$.

We now state a number of propositions:

(6) If f is strictly convex on X, then f is convex on X.

(7) Let a, b be real numbers and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then f is strictly convex on $[a,b]$ if and only if the following conditions are satisfied:

(i) $[a,b] \subseteq \operatorname{dom} f$, and

(ii) for every real number p such that $0 < p$ and $p < 1$ and for all real numbers r, s such that $r \in [a,b]$ and $s \in [a,b]$ and $r \neq s$ holds $f(p \cdot r + (1-p) \cdot s) < p \cdot f(r) + (1-p) \cdot f(s)$.

(8) Let X be a set and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then f is convex on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all real numbers a, b, c such that $a \in X$ and $b \in X$ and $c \in X$ and $a < b$ and $b < c$ holds $f(b) \leq \frac{c-b}{c-a} \cdot f(a) + \frac{b-a}{c-a} \cdot f(c)$.

(9) Let X be a set and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then f is strictly convex on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all real numbers a, b, c such that $a \in X$ and $b \in X$ and $c \in X$ and $a < b$ and $b < c$ holds $f(b) < \frac{c-b}{c-a} \cdot f(a) + \frac{b-a}{c-a} \cdot f(c)$.

(10) If f is strictly convex on X and $Y \subseteq X$, then f is strictly convex on Y.

(11) f is strictly convex on X iff $f - r$ is strictly convex on X.

(12) If $0 < r$, then f is strictly convex on X iff $r\,f$ is strictly convex on X.

(13) If f is strictly convex on X and g is strictly convex on X, then $f + g$ is strictly convex on X.

(14) If f is strictly convex on X and g is convex on X, then $f + g$ is strictly convex on X.

(15) Suppose f is strictly convex on X but g is strictly convex on X but $a > 0$ and $b \geq 0$ or $a \geq 0$ and $b > 0$. Then $a\,f + b\,g$ is strictly convex on X.

(16) f is convex on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all a, b, r such that $a \in X$ and $b \in X$ and $r \in X$ and $a < r$ and $r < b$ holds $\frac{f(r)-f(a)}{r-a} \leq \frac{f(b)-f(a)}{b-a}$ and $\frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(r)}{b-r}$.

(17) f is strictly convex on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all a, b, r such that $a \in X$ and $b \in X$ and $r \in X$ and $a < r$ and $r < b$ holds $\frac{f(r)-f(a)}{r-a} < \frac{f(b)-f(a)}{b-a}$ and $\frac{f(b)-f(a)}{b-a} < \frac{f(b)-f(r)}{b-r}$.

(18) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is total. Then for every natural number n and for all elements P, E, F of $\mathbb{R}^n$ such that $\sum P = 1$ and for every natural number i such that $i \in \operatorname{dom} P$ holds $P(i) \geq 0$ and $F(i) = f(E(i))$ holds $f(\sum(P \bullet E)) \leq \sum(P \bullet F)$ if and only if f is convex on $\mathbb{R}$.

(19) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, I be an interval, and a be a real number. Suppose there exist real numbers x_1, x_2 such that $x_1 \in I$ and $x_2 \in I$ and $x_1 < a$ and $a < x_2$ and f is convex on I. Then f is continuous in a.

3. Definitions of Several Convexity and Semicontinuity Concepts

Let us consider f, X. We say that f is quasiconvex on X if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) $X \subseteq \mathrm{dom}\, f$, and

(ii) for every real number p such that $0 < p$ and $p < 1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r + (1-p) \cdot s \in X$ holds $f(p \cdot r + (1-p) \cdot s) \leq \max(f(r), f(s))$.

Let us consider f, X. We say that f is strictly quasiconvex on X if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $X \subseteq \mathrm{dom}\, f$, and

(ii) for every real number p such that $0 < p$ and $p < 1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r + (1-p) \cdot s \in X$ and $f(r) \neq f(s)$ holds $f(p \cdot r + (1-p) \cdot s) < \max(f(r), f(s))$.

Let us consider f, X. We say that f is strongly quasiconvex on X if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) $X \subseteq \mathrm{dom}\, f$, and

(ii) for every real number p such that $0 < p$ and $p < 1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r + (1-p) \cdot s \in X$ and $r \neq s$ holds $f(p \cdot r + (1-p) \cdot s) < \max(f(r), f(s))$.

Let us consider f and let x_0 be a real number. We say that f is upper semicontinuous in x_0 if and only if:

(Def. 5) $x_0 \in \mathrm{dom}\, f$ and for every r such that $0 < r$ there exists s such that $0 < s$ and for every x such that $x \in \mathrm{dom}\, f$ and $|x - x_0| < s$ holds $f(x_0) - f(x) < r$.

Let us consider f, X. We say that f is upper semicontinuous on X if and only if:

(Def. 6) $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f \restriction X$ is upper semicontinuous in x_0.

Let us consider f and let x_0 be a real number. We say that f is lower semicontinuous in x_0 if and only if:

(Def. 7) $x_0 \in \mathrm{dom}\, f$ and for every r such that $0 < r$ there exists s such that $0 < s$ and for every x such that $x \in \mathrm{dom}\, f$ and $|x - x_0| < s$ holds $f(x) - f(x_0) < r$.

Let us consider f, X. We say that f is lower semicontinuous on X if and only if:

(Def. 8) $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f \restriction X$ is lower semicontinuous in x_0.

Next we state several propositions:

(20) Let x_0 be a real number and given f. Then f is upper semicontinuous in x_0 and f is lower semicontinuous in x_0 if and only if f is continuous in x_0.

(21) Let given X, f. Then f is upper semicontinuous on X and f is lower semicontinuous on X if and only if f is continuous on X.

(22) For all X, f such that f is strictly convex on X holds f is strongly quasiconvex on X.

(23) For all X, f such that f is strongly quasiconvex on X holds f is quasiconvex on X.

(24) For all X, f such that f is convex on X holds f is strictly quasiconvex on X.

(25) For all X, f such that f is strongly quasiconvex on X holds f is strictly quasiconvex on X.

(26) Let given X, f. Suppose f is strictly quasiconvex on X and f is one-to-one. Then f is strongly quasiconvex on X.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[3] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/supinf_1.html.

[4] Józef Białas. Properties of the intervals of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/measure5.html.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[6] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[7] Czesław Byliński. Binary operations applied to finite sequences. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseqop.html.

[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[9] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rvsum_1.html.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[11] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[12] Jarosław Kotowicz and Yuji Sakai. Properties of partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/rfunct_3.html.

[13] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[14] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/fcont_1.html.

[15] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rcomp_1.html.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[17] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/square_1.html.

[18] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[19] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received March 23, 2000

Published January 2, 2004