JOURNAL OF FORMALIZED MATHEMATICS
Volume 5, Released 1993, Published 2003
Inst. of Computer Science, Univ. of Białystok

Properties of Partial Functions from a Domain to the Set of Real Numbers

Jarosław Kotowicz
Warsaw University
Białystok

Yuji Sakai
Shinshu University
Nagano

Summary. The article consists of two parts. In the first one we consider notion of nonnegative and nonpositive part of a real numbers. In the second we consider partial function from a domain to the set of real numbers (or more general to a domain). We define a few new operations for these functions and show connections between finite sequences of real numbers and functions which domain is finite. We introduce *integrations* for finite domain real valued functions.

MML Identifier: RFUNCT_3.

WWW: http://mizar.org/JFM/Vol5/rfunct_3.html

The articles [22], [26], [2], [23], [27], [5], [3], [4], [1], [12], [18], [20], [21], [8], [24], [28], [6], [7], [13], [16], [25], [10], [9], [19], [15], [14], [11], and [17] provide the notation and terminology for this paper.

1. Nonnegative and Nonpositive Part of a Real Number

In this paper n denotes a natural number and r denotes a real number.

Let n, m be natural numbers. Then $\min(n,m)$ is a natural number.

Let r be a real number. The functor $\max_+(r)$ yields a real number and is defined as follows:

(Def. 1) $\max_+(r) = \max(r,0)$.

The functor $\max_-(r)$ yields a real number and is defined as follows:

(Def. 2) $\max_-(r) = \max(-r,0)$.

The following propositions are true:

(1) For every real number r holds $r = \max_+(r) - \max_-(r)$.

(2) For every real number r holds $|r| = \max_+(r) + \max_-(r)$.

(3) For every real number r holds $2 \cdot \max_+(r) = r + |r|$.

(4) For all real numbers r, s such that $0 \leq r$ holds $\max_+(r \cdot s) = r \cdot \max_+(s)$.

(5) For all real numbers r, s holds $\max_+(r+s) \leq \max_+(r) + \max_+(s)$.

(6) For every real number r holds $0 \leq \max_+(r)$ and $0 \leq \max_-(r)$.

(7) For all real numbers r_1, r_2, s_1, s_2 such that $r_1 \leq s_1$ and $r_2 \leq s_2$ holds $\max(r_1,r_2) \leq \max(s_1,s_2)$.

2. Properties of Real Function

The following propositions are true:

(8) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r, s be real numbers. If $r \neq 0$, then $F^{-1}(\{\frac{s}{r}\}) = (r\,F)^{-1}(\{s\})$.

(9) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $(0\,F)^{-1}(\{0\}) = \operatorname{dom} F$.

(10) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r be a real number. If $0 < r$, then $|F|^{-1}(\{r\}) = F^{-1}(\{-r, r\})$.

(11) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $|F|^{-1}(\{0\}) = F^{-1}(\{0\})$.

(12) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r be a real number. If $r < 0$, then $|F|^{-1}(\{r\}) = \emptyset$.

(13) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, G be a partial function from C to $\mathbb{R}$, and r be a real number. Suppose $r \neq 0$. Then F and G are fiberwise equipotent if and only if $r\,F$ and $r\,G$ are fiberwise equipotent.

(14) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, and G be a partial function from C to $\mathbb{R}$. Then F and G are fiberwise equipotent if and only if $-F$ and $-G$ are fiberwise equipotent.

(15) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, and G be a partial function from C to $\mathbb{R}$. Suppose F and G are fiberwise equipotent. Then $|F|$ and $|G|$ are fiberwise equipotent.

Let X, Y be sets. Set of partial functions from X to Y is defined by:

(Def. 3) Every element of it is a partial function from X to Y.

Let X, Y be sets. Note that there exists a set of partial functions from X to Y which is non empty.

Let X, Y be sets. A non empty set of partial functions from X to Y is a non empty set of partial functions from X to Y.

Let X, Y be sets. Then $X \dot{\to} Y$ is a set of partial functions from X to Y. Let P be a non empty set of partial functions from X to Y. We see that the element of P is a partial function from X to Y.

Let D, C be non empty sets, let X be a subset of D, and let c be an element of C. Then $X \longmapsto c$ is an element of $D \dot{\to} C$.

Let D be a non empty set and let F_1, F_2 be elements of $D \dot{\to} \mathbb{R}$. Then $F_1 + F_2$ is an element of $D \dot{\to} \mathbb{R}$. Then $F_1 - F_2$ is an element of $D \dot{\to} \mathbb{R}$. Then $F_1\,F_2$ is an element of $D \dot{\to} \mathbb{R}$. Then $\frac{F_1}{F_2}$ is an element of $D \dot{\to} \mathbb{R}$.

Let D be a non empty set and let F be an element of $D \dot{\to} \mathbb{R}$. Then $|F|$ is an element of $D \dot{\to} \mathbb{R}$. Then $-F$ is an element of $D \dot{\to} \mathbb{R}$. Then $\frac{1}{F}$ is an element of $D \dot{\to} \mathbb{R}$.

Let D be a non empty set, let F be an element of $D \dot{\to} \mathbb{R}$, and let r be a real number. Then $r\,F$ is an element of $D \dot{\to} \mathbb{R}$.

Let D be a non empty set. The functor $+_{D \dot{\to} \mathbb{R}}$ yielding a binary operation on $D \dot{\to} \mathbb{R}$ is defined as follows:

(Def. 4) For all elements F_1, F_2 of $D \dot{\to} \mathbb{R}$ holds $+_{D \dot{\to} \mathbb{R}}(F_1, F_2) = F_1 + F_2$.

One can prove the following propositions:

(16) For every non empty set D holds $+_{D \dot{\to} \mathbb{R}}$ is commutative.

(17) For every non empty set D holds $+_{D \dot{\to} \mathbb{R}}$ is associative.

(18) For every non empty set D holds $\Omega_D \longmapsto (0$ **qua** real number$)$ is a unity w.r.t. $+_{D \dot{\to} \mathbb{R}}$.

(19) For every non empty set D holds $\mathbf{1}_{+_{D \dot\rightarrow \mathbb{R}}} = \Omega_D \longmapsto (0$ **qua** real number$)$.

(20) For every non empty set D holds $+_{D \dot\rightarrow \mathbb{R}}$ has a unity.

Let D be a non empty set and let f be a finite sequence of elements of $D \dot\rightarrow \mathbb{R}$. The functor $\sum f$ yields an element of $D \dot\rightarrow \mathbb{R}$ and is defined by:

(Def. 5) $\sum f = +_{D \dot\rightarrow \mathbb{R}} \circledast f$.

One can prove the following propositions:

(21) For every non empty set D holds $\sum(\varepsilon_{(D \dot\rightarrow \mathbb{R})}) = \Omega_D \longmapsto (0$ **qua** real number$)$.

(22) For every non empty set D and for every element G of $D \dot\rightarrow \mathbb{R}$ holds $\sum\langle G\rangle = G$.

(23) Let D be a non empty set, f be a finite sequence of elements of $D \dot\rightarrow \mathbb{R}$, and G be an element of $D \dot\rightarrow \mathbb{R}$. Then $\sum(f \frown \langle G\rangle) = \sum f + G$.

(24) For every non empty set D and for all finite sequences f_1, f_2 of elements of $D \dot\rightarrow \mathbb{R}$ holds $\sum(f_1 \frown f_2) = \sum f_1 + \sum f_2$.

(25) Let D be a non empty set, f be a finite sequence of elements of $D \dot\rightarrow \mathbb{R}$, and G be an element of $D \dot\rightarrow \mathbb{R}$. Then $\sum(\langle G\rangle \frown f) = G + \sum f$.

(26) For every non empty set D and for all elements G_1, G_2 of $D \dot\rightarrow \mathbb{R}$ holds $\sum\langle G_1, G_2\rangle = G_1 + G_2$.

(27) For every non empty set D and for all elements G_1, G_2, G_3 of $D \dot\rightarrow \mathbb{R}$ holds $\sum\langle G_1, G_2, G_3\rangle = G_1 + G_2 + G_3$.

(28) Let D be a non empty set and f, g be finite sequences of elements of $D \dot\rightarrow \mathbb{R}$. If f and g are fiberwise equipotent, then $\sum f = \sum g$.

Let D be a non empty set and let f be a finite sequence. The functor $\mathrm{CHI}(f, D)$ yields a finite sequence of elements of $D \dot\rightarrow \mathbb{R}$ and is defined as follows:

(Def. 6) $\mathrm{len}\,\mathrm{CHI}(f, D) = \mathrm{len}\, f$ and for every n such that $n \in \mathrm{dom}\,\mathrm{CHI}(f, D)$ holds $(\mathrm{CHI}(f, D))(n) = \chi_{f(n), D}$.

Let D be a non empty set, let f be a finite sequence of elements of $D \dot\rightarrow \mathbb{R}$, and let R be a finite sequence of elements of $\mathbb{R}$. The functor $R\, f$ yielding a finite sequence of elements of $D \dot\rightarrow \mathbb{R}$ is defined by the conditions (Def. 7).

(Def. 7)(i) $\mathrm{len}(R\, f) = \min(\mathrm{len}\, R, \mathrm{len}\, f)$, and

(ii) for every n such that $n \in \mathrm{dom}(R\, f)$ and for every partial function F from D to $\mathbb{R}$ and for every r such that $r = R(n)$ and $F = f(n)$ holds $(R\, f)(n) = r\, F$.

Let D be a non empty set, let f be a finite sequence of elements of $D \dot\rightarrow \mathbb{R}$, and let d be an element of D. The functor $f\#d$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by the conditions (Def. 8).

(Def. 8)(i) $\mathrm{len}(f\#d) = \mathrm{len}\, f$, and

(ii) for every natural number n and for every element G of $D \dot\rightarrow \mathbb{R}$ such that $n \in \mathrm{dom}(f\#d)$ and $f(n) = G$ holds $(f\#d)(n) = G(d)$.

Let D, C be non empty sets, let f be a finite sequence of elements of $D \dot\rightarrow C$, and let d be an element of D. We say that d is common for dom f if and only if:

(Def. 9) For every element G of $D \dot\rightarrow C$ and for every natural number n such that $n \in \mathrm{dom}\, f$ and $f(n) = G$ holds $d \in \mathrm{dom}\, G$.

The following propositions are true:

(29) Let D, C be non empty sets, f be a finite sequence of elements of $D \dot{\to} C$, d be an element of D, and n be a natural number. If d is common for dom f and $n \neq 0$, then d is common for dom $f \restriction n$.

(30) Let D, C be non empty sets, f be a finite sequence of elements of $D \dot{\to} C$, d be an element of D, and n be a natural number. If d is common for dom f, then d is common for dom $f_{\lfloor n}$.

(31) Let D be a non empty set, d be an element of D, and f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$. If len $f \neq 0$, then d is common for dom f iff $d \in \mathrm{dom} \sum f$.

(32) Let D be a non empty set, f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$, d be an element of D, and n be a natural number. Then $(f \restriction n)\#d = (f\#d) \restriction n$.

(33) For every non empty set D and for every finite sequence f and for every element d of D holds d is common for dom $\mathrm{CHI}(f, D)$.

(34) Let D be a non empty set, d be an element of D, f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$, and R be a finite sequence of elements of $\mathbb{R}$. If d is common for dom f, then d is common for dom $R\, f$.

(35) Let D be a non empty set, f be a finite sequence, R be a finite sequence of elements of $\mathbb{R}$, and d be an element of D. Then d is common for dom $R\,\mathrm{CHI}(f, D)$.

(36) Let D be a non empty set, d be an element of D, and f be a finite sequence of elements of $D \dot{\to} \mathbb{R}$. If d is common for dom f, then $(\sum f)(d) = \sum(f\#d)$.

Let D be a non empty set and let F be a partial function from D to $\mathbb{R}$. The functor $\max_+(F)$ yields a partial function from D to $\mathbb{R}$ and is defined by:

(Def. 10) $\mathrm{dom}\max_+(F) = \mathrm{dom} F$ and for every element d of D such that $d \in \mathrm{dom}\max_+(F)$ holds $(\max_+(F))(d) = \max_+(F(d))$.

The functor $\max_-(F)$ yielding a partial function from D to $\mathbb{R}$ is defined by:

(Def. 11) $\mathrm{dom}\max_-(F) = \mathrm{dom} F$ and for every element d of D such that $d \in \mathrm{dom}\max_-(F)$ holds $(\max_-(F))(d) = \max_-(F(d))$.

The following propositions are true:

(37) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $F = \max_+(F) - \max_-(F)$ and $|F| = \max_+(F) + \max_-(F)$ and $2\, \max_+(F) = F + |F|$.

(38) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r be a real number. If $0 < r$, then $F^{-1}(\{r\}) = (\max_+(F))^{-1}(\{r\})$.

(39) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $F^{-1}(]-\infty, 0]) = (\max_+(F))^{-1}(\{0\})$.

(40) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and d be an element of D. If $d \in \mathrm{dom} F$, then $0 \leq (\max_+(F))(d)$.

(41) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r be a real number. If $0 < r$, then $F^{-1}(\{-r\}) = (\max_-(F))^{-1}(\{r\})$.

(42) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $F^{-1}([0, +\infty[) = (\max_-(F))^{-1}(\{0\})$.

(43) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and d be an element of D. If $d \in \mathrm{dom} F$, then $0 \leq (\max_-(F))(d)$.

(44) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, and G be a partial function from C to $\mathbb{R}$. Suppose F and G are fiberwise equipotent. Then $\max_+(F)$ and $\max_+(G)$ are fiberwise equipotent.

(45) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, and G be a partial function from C to $\mathbb{R}$. Suppose F and G are fiberwise equipotent. Then $\max_-(F)$ and $\max_-(G)$ are fiberwise equipotent.

Let A, B be sets. Note that there exists a partial function from A to B which is finite.

Let D be a non empty set and let F be a finite partial function from D to $\mathbb{R}$. One can check that $\max_+(F)$ is finite and $\max_-(F)$ is finite.

Next we state several propositions:

(46) Let D, C be non empty sets, F be a finite partial function from D to $\mathbb{R}$, and G be a finite partial function from C to $\mathbb{R}$. Suppose $\max_+(F)$ and $\max_+(G)$ are fiberwise equipotent and $\max_-(F)$ and $\max_-(G)$ are fiberwise equipotent. Then F and G are fiberwise equipotent.

(47) For every non empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X holds $\max_+(F){\restriction}X = \max_+(F{\restriction}X)$.

(48) For every non empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X holds $\max_-(F){\restriction}X = \max_-(F{\restriction}X)$.

(49) Let D be a non empty set and F be a partial function from D to $\mathbb{R}$. If for every element d of D such that $d \in \mathrm{dom}\, F$ holds $F(d) \geq 0$, then $\max_+(F) = F$.

(50) Let D be a non empty set and F be a partial function from D to $\mathbb{R}$. If for every element d of D such that $d \in \mathrm{dom}\, F$ holds $F(d) \leq 0$, then $\max_-(F) = -F$.

Let D be a non empty set, let F be a partial function from D to $\mathbb{R}$, and let r be a real number. The functor $F - r$ yielding a partial function from D to $\mathbb{R}$ is defined as follows:

(Def. 12) $\mathrm{dom}(F - r) = \mathrm{dom}\, F$ and for every element d of D such that $d \in \mathrm{dom}(F - r)$ holds $(F - r)(d) = F(d) - r$.

Next we state four propositions:

(51) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $F - 0 = F$.

(52) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, r be a real number, and X be a set. Then $F{\restriction}X - r = (F - r){\restriction}X$.

(53) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and r, s be real numbers. Then $F^{-1}(\{s + r\}) = (F - r)^{-1}(\{s\})$.

(54) Let D, C be non empty sets, F be a partial function from D to $\mathbb{R}$, G be a partial function from C to $\mathbb{R}$, and r be a real number. Then F and G are fiberwise equipotent if and only if $F - r$ and $G - r$ are fiberwise equipotent.

Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let X be a set. We say that F is convex on X if and only if the conditions (Def. 13) are satisfied.

(Def. 13)(i) $X \subseteq \mathrm{dom}\, F$, and

(ii) for every real number p such that $0 \leq p$ and $p \leq 1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r + (1 - p) \cdot s \in X$ holds $F(p \cdot r + (1 - p) \cdot s) \leq p \cdot F(r) + (1 - p) \cdot F(s)$.

The following propositions are true:

(55) Let a, b be real numbers and F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then F is convex on $[a, b]$ if and only if the following conditions are satisfied:

(i) $[a, b] \subseteq \mathrm{dom}\, F$, and

(ii) for every real number p such that $0 \leq p$ and $p \leq 1$ and for all real numbers r, s such that $r \in [a, b]$ and $s \in [a, b]$ holds $F(p \cdot r + (1 - p) \cdot s) \leq p \cdot F(r) + (1 - p) \cdot F(s)$.

(56) Let a, b be real numbers and F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then F is convex on $[a,b]$ if and only if the following conditions are satisfied:

(i) $[a,b] \subseteq \mathrm{dom}\, F$, and

(ii) for all real numbers x_1, x_2, x_3 such that $x_1 \in [a,b]$ and $x_2 \in [a,b]$ and $x_3 \in [a,b]$ and $x_1 < x_2$ and $x_2 < x_3$ holds $\frac{F(x_1)-F(x_2)}{x_1-x_2} \leq \frac{F(x_2)-F(x_3)}{x_2-x_3}$.

(57) Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and X, Y be sets. If F is convex on X and $Y \subseteq X$, then F is convex on Y.

(58) Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$, X be a set, and r be a real number. Then F is convex on X if and only if $F - r$ is convex on X.

(59) Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$, X be a set, and r be a real number. Suppose $0 < r$. Then F is convex on X if and only if $r\, F$ is convex on X.

(60) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X such that $X \subseteq \mathrm{dom}\, F$ holds $0\, F$ is convex on X.

(61) Let F, G be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and X be a set. Suppose F is convex on X and G is convex on X. Then $F + G$ is convex on X.

(62) Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$, X be a set, and r be a real number. If F is convex on X, then $\max_+(F - r)$ is convex on X.

(63) Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and X be a set. If F is convex on X, then $\max_+(F)$ is convex on X.

(64) $\mathrm{id}_{\Omega_{\mathbb{R}}}$ is convex on $\mathbb{R}$.

(65) For every real number r holds $\max_+(\mathrm{id}_{\Omega_{\mathbb{R}}} - r)$ is convex on $\mathbb{R}$.

Let D be a non empty set, let F be a partial function from D to $\mathbb{R}$, and let X be a set. Let us assume that $\mathrm{dom}(F{\restriction}X)$ is finite. The functor $\mathrm{FinS}(F,X)$ yielding a non-increasing finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 14) $F{\restriction}X$ and $\mathrm{FinS}(F,X)$ are fiberwise equipotent.

We now state two propositions:

(66) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X be a set. If $\mathrm{dom}(F{\restriction}X)$ is finite, then $\mathrm{FinS}(F, \mathrm{dom}(F{\restriction}X)) = \mathrm{FinS}(F,X)$.

(67) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X be a set. If $\mathrm{dom}(F{\restriction}X)$ is finite, then $\mathrm{FinS}(F{\restriction}X,X) = \mathrm{FinS}(F,X)$.

Let D be a non empty set, let F be a partial function from D to $\mathbb{R}$, and let X be a finite set. Then $F{\restriction}X$ is a finite partial function from D to $\mathbb{R}$.

The following propositions are true:

(68) Let D be a non empty set, d be an element of D, X be a set, and F be a partial function from D to $\mathbb{R}$. Suppose X is finite and $d \in \mathrm{dom}(F{\restriction}X)$. Then $(\mathrm{FinS}(F, X \setminus \{d\})) \frown \langle F(d) \rangle$ and $F{\restriction}X$ are fiberwise equipotent.

(69) Let D be a non empty set, d be an element of D, X be a set, and F be a partial function from D to $\mathbb{R}$. Suppose $\mathrm{dom}(F{\restriction}X)$ is finite and $d \in \mathrm{dom}(F{\restriction}X)$. Then $(\mathrm{FinS}(F, X \setminus \{d\})) \frown \langle F(d) \rangle$ and $F{\restriction}X$ are fiberwise equipotent.

(70) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, X be a set, and Y be a finite set. If $Y = \mathrm{dom}(F{\restriction}X)$, then $\mathrm{len}\, \mathrm{FinS}(F,X) = \mathrm{card}\, Y$.

(71) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $\mathrm{FinS}(F,\emptyset)=\varepsilon_{\mathbb{R}}$.

(72) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and d be an element of D. If $d\in\mathrm{dom}\,F$, then $\mathrm{FinS}(F,\{d\})=\langle F(d)\rangle$.

(73) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, X be a set, and d be an element of D. If $\mathrm{dom}(F{\restriction}X)$ is finite and $d\in\mathrm{dom}(F{\restriction}X)$ and $(\mathrm{FinS}(F,X))(\mathrm{len}\,\mathrm{FinS}(F,X))=F(d)$, then $\mathrm{FinS}(F,X)=(\mathrm{FinS}(F,X\setminus\{d\}))^\frown\langle F(d)\rangle$.

(74) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X, Y be sets. Suppose $\mathrm{dom}(F{\restriction}X)$ is finite and $Y\subseteq X$ and for all elements d_1, d_2 of D such that $d_1\in\mathrm{dom}(F{\restriction}Y)$ and $d_2\in\mathrm{dom}(F{\restriction}(X\setminus Y))$ holds $F(d_1)\geq F(d_2)$. Then $\mathrm{FinS}(F,X)=(\mathrm{FinS}(F,Y))^\frown\mathrm{FinS}(F,X\setminus Y)$.

(75) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, r be a real number, X be a set, and d be an element of D. Suppose $\mathrm{dom}(F{\restriction}X)$ is finite and $d\in\mathrm{dom}(F{\restriction}X)$. Then $(\mathrm{FinS}(F-r,X))(\mathrm{len}\,\mathrm{FinS}(F-r,X))=(F-r)(d)$ if and only if $(\mathrm{FinS}(F,X))(\mathrm{len}\,\mathrm{FinS}(F,X))=F(d)$.

(76) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, r be a real number, X be a set, and Z be a finite set. If $Z=\mathrm{dom}(F{\restriction}X)$, then $\mathrm{FinS}(F-r,X)=\mathrm{FinS}(F,X)-\mathrm{card}\,Z\mapsto r$.

(77) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X be a set. Suppose $\mathrm{dom}(F{\restriction}X)$ is finite and for every element d of D such that $d\in\mathrm{dom}(F{\restriction}X)$ holds $F(d)\geq 0$. Then $\mathrm{FinS}(\max_+(F),X)=\mathrm{FinS}(F,X)$.

(78) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, X be a set, r be a real number, and Z be a finite set. If $Z=\mathrm{dom}(F{\restriction}X)$ and $\mathrm{rng}(F{\restriction}X)=\{r\}$, then $\mathrm{FinS}(F,X)=\mathrm{card}\,Z\mapsto r$.

(79) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X, Y be sets. Suppose $\mathrm{dom}(F{\restriction}(X\cup Y))$ is finite and X misses Y. Then $\mathrm{FinS}(F,X\cup Y)$ and $(\mathrm{FinS}(F,X))^\frown\mathrm{FinS}(F,Y)$ are fiberwise equipotent.

Let D be a non empty set, let F be a partial function from D to $\mathbb{R}$, and let X be a set. The functor $\sum_{\kappa=0}^{X}F(\kappa)$ yielding a real number is defined by:

(Def. 15) $\sum_{\kappa=0}^{X}F(\kappa)=\sum\mathrm{FinS}(F,X)$.

Next we state several propositions:

(80) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, X be a set, and r be a real number. If $\mathrm{dom}(F{\restriction}X)$ is finite, then $\sum_{\kappa=0}^{X}(r\,F)(\kappa)=r\cdot\sum_{\kappa=0}^{X}F(\kappa)$.

(81) Let D be a non empty set, F, G be partial functions from D to $\mathbb{R}$, X be a set, and Y be a finite set. If $Y=\mathrm{dom}(F{\restriction}X)$ and $\mathrm{dom}(F{\restriction}X)=\mathrm{dom}(G{\restriction}X)$, then $\sum_{\kappa=0}^{X}(F+G)(\kappa)=\sum_{\kappa=0}^{X}F(\kappa)+\sum_{\kappa=0}^{X}G(\kappa)$.

(82) Let D be a non empty set, F, G be partial functions from D to $\mathbb{R}$, and X be a set. If $\mathrm{dom}(F{\restriction}X)$ is finite and $\mathrm{dom}(F{\restriction}X)=\mathrm{dom}(G{\restriction}X)$, then $\sum_{\kappa=0}^{X}(F-G)(\kappa)=\sum_{\kappa=0}^{X}F(\kappa)-\sum_{\kappa=0}^{X}G(\kappa)$.

(83) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, X be a set, r be a real number, and Y be a finite set. If $Y=\mathrm{dom}(F{\restriction}X)$, then $\sum_{\kappa=0}^{X}(F-r)(\kappa)=\sum_{\kappa=0}^{X}F(\kappa)-r\cdot\mathrm{card}\,Y$.

(84) For every non empty set D and for every partial function F from D to $\mathbb{R}$ holds $\sum_{\kappa=0}^{\emptyset}F(\kappa)=0$.

(85) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and d be an element of D. If $d\in\mathrm{dom}\,F$, then $\sum_{\kappa=0}^{\{d\}}F(\kappa)=F(d)$.

(86) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X, Y be sets. If $\mathrm{dom}(F{\restriction}(X\cup Y))$ is finite and X misses Y, then $\sum_{\kappa=0}^{X\cup Y}F(\kappa)=\sum_{\kappa=0}^{X}F(\kappa)+\sum_{\kappa=0}^{Y}F(\kappa)$.

(87) Let D be a non empty set, F be a partial function from D to $\mathbb{R}$, and X, Y be sets. If $\mathrm{dom}(F{\restriction}(X\cup Y))$ is finite and $\mathrm{dom}(F{\restriction}X)$ misses $\mathrm{dom}(F{\restriction}Y)$, then $\sum_{\kappa=0}^{X\cup Y}F(\kappa)=\sum_{\kappa=0}^{X}F(\kappa)+\sum_{\kappa=0}^{Y}F(\kappa)$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[4] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[6] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[7] Czesław Byliński. Binary operations applied to finite sequences. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseqop.html.

[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[9] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rvsum_1.html.

[10] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_T$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[13] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[14] Jarosław Kotowicz. The limit of a real function at infinity. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/limfunc1.html.

[15] Jarosław Kotowicz. Partial functions from a domain to a domain. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/partfun2.html.

[16] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rfunct_1.html.

[17] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/rfinseq.html.

[18] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[19] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rcomp_1.html.

[20] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[21] Andrzej Trybulec. Semilattice operations on finite subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setwiseo.html.

[22] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[23] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[24] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/square_1.html.

[25] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[26] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[27] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[28] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received March 15, 1993

Published January 2, 2004