JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Several Properties of Fields. Field Theory

Józef Białas
University of Łódź

Summary. The article includes a continuation of the paper [1]. Some simple theorems concerning basic properties of a field are proved.

MML Identifier: REALSET3.

WWW: http://mizar.org/JFM/Vol2/realset3.html

The articles [5], [4], [6], [2], [3], and [1] provide the notation and terminology for this paper.

The following propositions are true:

(1) For every field F holds $-_F(\mathbf{0}_F) = \mathbf{0}_F$.

(2) For every field F holds $(^{-1}_F)(\mathbf{1}_F) = \mathbf{1}_F$.

(3) For every field F and for all elements a, b of the support of F holds $-_F(+_F(\langle a, -_F(b)\rangle)) = +_F(\langle b, -_F(a)\rangle)$.

(4) For every field F and for all elements a, b of (the support of $F) \setminus \{\mathbf{0}_F\}$ holds $(^{-1}_F)(\cdot_F(\langle a, (^{-1}_F)(b)\rangle)) = \cdot_F(\langle b, (^{-1}_F)(a)\rangle)$.

(5) For every field F and for all elements a, b of the support of F holds $-_F(+_F(\langle a, b\rangle)) = +_F(\langle -_F(a), -_F(b)\rangle)$.

(6) For every field F and for all elements a, b of (the support of $F) \setminus \{\mathbf{0}_F\}$ holds $(^{-1}_F)(\cdot_F(\langle a, b\rangle)) = \cdot_F(\langle (^{-1}_F)(a), (^{-1}_F)(b)\rangle)$.

(7) Let F be a field and a, b, c, d be elements of the support of F. Then $+_F(\langle a, -_F(b)\rangle) = +_F(\langle c, -_F(d)\rangle)$ if and only if $+_F(\langle a, d\rangle) = +_F(\langle b, c\rangle)$.

(8) Let F be a field, a, c be elements of the support of F, and b, d be elements of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $\cdot_F(\langle a, (^{-1}_F)(b)\rangle) = \cdot_F(\langle c, (^{-1}_F)(d)\rangle)$ if and only if $\cdot_F(\langle a, d\rangle) = \cdot_F(\langle b, c\rangle)$.

(9) For every field F and for all elements a, b of the support of F holds $\cdot_F(\langle a, b\rangle) = \mathbf{0}_F$ iff $a = \mathbf{0}_F$ or $b = \mathbf{0}_F$.

(10) Let F be a field, a, b be elements of the support of F, and c, d be elements of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $\cdot_F(\langle \cdot_F(\langle a, (^{-1}_F)(c)\rangle), \cdot_F(\langle b, (^{-1}_F)(d)\rangle)\rangle) = \cdot_F(\langle \cdot_F(\langle a, b\rangle), (^{-1}_F)(\cdot_F(\langle c, d\rangle))\rangle)$.

(11) Let F be a field, a, b be elements of the support of F, and c, d be elements of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $+_F(\langle \cdot_F(\langle a, (^{-1}_F)(c)\rangle), \cdot_F(\langle b, (^{-1}_F)(d)\rangle)\rangle) = \cdot_F(\langle +_F(\langle \cdot_F(\langle a, d\rangle), \cdot_F(\langle b, c\rangle)\rangle), (^{-1}_F)(\cdot_F(\langle c, d\rangle))\rangle)$.

Let F be a field. The functor $\mathrm{osf}\,F$ yielding a binary operation on the support of F is defined by:

(Def. 1) For all elements x, y of the support of F holds $(\mathrm{osf}\,F)(\langle x, y\rangle) = +_F(\langle x, -_F(y)\rangle)$.

Next we state a number of propositions:

(14)[1] For every field F and for every element x of the support of F holds $(\mathrm{osf}F)(\langle x, x\rangle) = \mathbf{0}_F$.

(15) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle a, (\mathrm{osf}F)(\langle b, c\rangle)\rangle) = (\mathrm{osf}F)(\langle \cdot_F(\langle a, b\rangle), \cdot_F(\langle a, c\rangle)\rangle)$.

(16) Let F be a field and a, b be elements of the support of F. Then $(\mathrm{osf}F)(\langle a, b\rangle)$ is an element of the support of F.

(17) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle (\mathrm{osf}F)(\langle a, b\rangle), c\rangle) = (\mathrm{osf}F)(\langle \cdot_F(\langle a, c\rangle), \cdot_F(\langle b, c\rangle)\rangle)$.

(18) For every field F and for all elements a, b of the support of F holds $(\mathrm{osf}F)(\langle a, b\rangle) = -_F((\mathrm{osf}F)(\langle b, a\rangle))$.

(19) For every field F and for all elements a, b of the support of F holds $(\mathrm{osf}F)(\langle -_F(a), b\rangle) = -_F(+_F(\langle a, b\rangle))$.

(20) Let F be a field and a, b, c, d be elements of the support of F. Then $(\mathrm{osf}F)(\langle a, b\rangle) = (\mathrm{osf}F)(\langle c, d\rangle)$ if and only if $+_F(\langle a, d\rangle) = +_F(\langle b, c\rangle)$.

(21) For every field F and for every element a of the support of F holds $(\mathrm{osf}F)(\langle \mathbf{0}_F, a\rangle) = -_F(a)$.

(22) For every field F and for every element a of the support of F holds $(\mathrm{osf}F)(\langle a, \mathbf{0}_F\rangle) = a$.

(23) For every field F and for all elements a, b, c of the support of F holds $+_F(\langle a, b\rangle) = c$ iff $(\mathrm{osf}F)(\langle c, a\rangle) = b$.

(24) For every field F and for all elements a, b, c of the support of F holds $+_F(\langle a, b\rangle) = c$ iff $(\mathrm{osf}F)(\langle c, b\rangle) = a$.

(25) For every field F and for all elements a, b, c of the support of F holds $(\mathrm{osf}F)(\langle a, (\mathrm{osf}F)(\langle b, c\rangle)\rangle) = +_F(\langle (\mathrm{osf}F)(\langle a, b\rangle), c\rangle)$.

(26) For every field F and for all elements a, b, c of the support of F holds $(\mathrm{osf}F)(\langle a, +_F(\langle b, c\rangle)\rangle) = (\mathrm{osf}F)(\langle (\mathrm{osf}F)(\langle a, b\rangle), c\rangle)$.

Let F be a field. The functor $\mathrm{ovf}F$ yields a function from $[\!:$the support of F, (the support of $F) \setminus \{\mathbf{0}_F\}:\!]$ into the support of F and is defined by the condition (Def. 2).

(Def. 2) Let x be an element of the support of F and y be an element of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $(\mathrm{ovf}F)(\langle x, y\rangle) = \cdot_F(\langle x, (^{-1}_F)(y)\rangle)$.

We now state a number of propositions:

(29)[2] For every field F and for every element x of (the support of $F) \setminus \{\mathbf{0}_F\}$ holds $(\mathrm{ovf}F)(\langle x, x\rangle) = \mathbf{1}_F$.

(30) Let F be a field, a be an element of the support of F, and b be an element of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $(\mathrm{ovf}F)(\langle a, b\rangle)$ is an element of the support of F.

(31) Let F be a field, a, b be elements of the support of F, and c be an element of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $\cdot_F(\langle a, (\mathrm{ovf}F)(\langle b, c\rangle)\rangle) = (\mathrm{ovf}F)(\langle \cdot_F(\langle a, b\rangle), c\rangle)$.

(32) For every field F and for every element a of (the support of $F) \setminus \{\mathbf{0}_F\}$ holds $\cdot_F(\langle a, (\mathrm{ovf}F)(\langle \mathbf{1}_F, a\rangle)\rangle) = \mathbf{1}_F$ and $\cdot_F(\langle (\mathrm{ovf}F)(\langle \mathbf{1}_F, a\rangle), a\rangle) = \mathbf{1}_F$.

[1] The propositions (12) and (13) have been removed.
[2] The propositions (27) and (28) have been removed.

(35)[3] For every field F and for all elements a, b of (the support of $F) \setminus \{\mathbf{0}_F\}$ holds $(\mathrm{ovf}F)(\langle a, b\rangle) = (^{-1}_F)((\mathrm{ovf}F)(\langle b, a\rangle))$.

(36) For every field F and for all elements a, b of (the support of $F) \setminus \{\mathbf{0}_F\}$ holds $(\mathrm{ovf}F)(\langle(^{-1}_F)(a), b\rangle) = (^{-1}_F)(\cdot_F(\langle a, b\rangle))$.

(37) Let F be a field, a, c be elements of the support of F, and b, d be elements of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $(\mathrm{ovf}F)(\langle a, b\rangle) = (\mathrm{ovf}F)(\langle c, d\rangle)$ if and only if $\cdot_F(\langle a, d\rangle) = \cdot_F(\langle b, c\rangle)$.

(38) For every field F and for every element a of (the support of $F) \setminus \{\mathbf{0}_F\}$ holds $(\mathrm{ovf}F)(\langle \mathbf{1}_F, a\rangle) = (^{-1}_F)(a)$.

(39) For every field F and for every element a of the support of F holds $(\mathrm{ovf}F)(\langle a, \mathbf{1}_F\rangle) = a$.

(40) Let F be a field, a be an element of (the support of $F) \setminus \{\mathbf{0}_F\}$, and b, c be elements of the support of F. Then $\cdot_F(\langle a, b\rangle) = c$ if and only if $(\mathrm{ovf}F)(\langle c, a\rangle) = b$.

(41) Let F be a field, a, c be elements of the support of F, and b be an element of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $\cdot_F(\langle a, b\rangle) = c$ if and only if $(\mathrm{ovf}F)(\langle c, b\rangle) = a$.

(42) Let F be a field, a be an element of the support of F, and b, c be elements of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $(\mathrm{ovf}F)(\langle a, (\mathrm{ovf}F)(\langle b, c\rangle)\rangle) = \cdot_F(\langle(\mathrm{ovf}F)(\langle a, b\rangle), c\rangle)$.

(43) Let F be a field, a be an element of the support of F, and b, c be elements of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $(\mathrm{ovf}F)(\langle a, \cdot_F(\langle b, c\rangle)\rangle) = (\mathrm{ovf}F)(\langle(\mathrm{ovf}F)(\langle a, b\rangle), c\rangle)$.

REFERENCES

[1] Józef Białas. Properties of fields. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/realset2.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[4] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[6] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

Received September 27, 1990

Published January 2, 2004

---

[3] The propositions (33) and (34) have been removed.