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Group and Field Definitions

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Summary. The article contains exactly the same definitions of group and field as those in [4]. These definitions were prepared without the help of the definitions and properties of *Nat* and *Real* modes included in the MML. This is the first of a series of articles in which we are going to introduce the concept of the set of real numbers in a elementary axiomatic way.

MML Identifier: REALSET1.

WWW: http://mizar.org/JFM/Vol1/realset1.html

The articles [6], [3], [8], [9], [1], [2], [7], and [5] provide the notation and terminology for this paper.

The following two propositions are true:

(10)[1] For all sets X, x and for every function F from $[:X, X:]$ into X such that $x \in [:X, X:]$ holds $F(x) \in X$.

(11) Let X be a set and F be a binary operation on X. Then there exists a subset A of X such that for every set x such that $x \in [:A, A:]$ holds $F(x) \in A$.

Let X be a set, let F be a binary operation on X, and let A be a subset of X. We say that F is in A if and only if:

(Def. 1) For every set x such that $x \in [:A, A:]$ holds $F(x) \in A$.

Let X be a set and let F be a binary operation on X. A subset of X is called a set closed w.r.t. F if:

(Def. 2) For every set x such that $x \in [:\mathrm{it}, \mathrm{it}:]$ holds $F(x) \in \mathrm{it}$.

One can prove the following proposition

(14)[2] Let X be a set, F be a binary operation on X, and A be a set closed w.r.t. F. Then $F{\restriction}[:A, A:]$ is a binary operation on A.

Let X be a set, let F be a binary operation on X, and let A be a set closed w.r.t. F. The functor $F \restriction A$ yields a binary operation on A and is defined by:

(Def. 3) $F \restriction A = F{\restriction}[:A, A:]$.

Let I_1 be a loop structure. We say that I_1 is zeroed if and only if the condition (Def. 5) is satisfied.

[1] The propositions (1)–(9) have been removed.
[2] The propositions (12) and (13) have been removed.

(Def. 5)[3] Let a be an element of I_1. Then (the addition of I_1)($\langle a$, the zero of $I_1\rangle$) $= a$ and (the addition of I_1)($\langle$the zero of I_1, $a\rangle$) $= a$.

We say that I_1 is complementable if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let a be an element of I_1. Then there exists an element b of I_1 such that (the addition of I_1)($\langle a, b\rangle$) $=$ the zero of I_1 and (the addition of I_1)($\langle b, a\rangle$) $=$ the zero of I_1.

Let L be a non empty loop structure. Let us observe that L is add-associative if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let a, b, c be elements of L. Then (the addition of L)($\langle$(the addition of L)($\langle a, b\rangle$), $c\rangle$) $=$ (the addition of L)($\langle a$, (the addition of L)($\langle b, c\rangle$)$\rangle$).

Let us observe that L is Abelian if and only if:

(Def. 8) For all elements a, b of L holds (the addition of L)($\langle a, b\rangle$) $=$ (the addition of L)($\langle b, a\rangle$).

Let X be a non empty set, let a be a binary operation on X, and let Z be an element of X. One can verify that $\langle X, a, Z\rangle$ is non empty.

Let us observe that there exists a non empty loop structure which is strict, Abelian, add-associative, zeroed, and complementable.

A group is an Abelian add-associative zeroed complementable non empty loop structure.

Let I_1 be a set. We say that I_1 is trivial if and only if:

(Def. 12)[4] I_1 is empty or there exists a set x such that $I_1 = \{x\}$.

One can verify the following observations:

* there exists a set which is trivial and non empty,
* there exists a set which is non trivial and non empty, and
* every set which is non trivial is also non empty.

We now state three propositions:

(32)[5] For every non empty set X holds X is non trivial iff for every set x holds $X \setminus \{x\}$ is a non empty set.

(33) There exists a non empty set A such that for every element z of A holds $A \setminus \{z\}$ is a non empty set.

(34) For every non empty set X such that for every element x of X holds $X \setminus \{x\}$ is a non empty set holds X is non trivial.

Let I_1 be a 1-sorted structure. We say that I_1 is trivial if and only if:

(Def. 13) The carrier of I_1 is trivial.

Let us observe that there exists a 1-sorted structure which is trivial.

Next we state the proposition

(35) For every non empty set A such that for every element x of A holds $A \setminus \{x\}$ is a non empty set holds A is a non trivial set.

Let us note that there exists a double loop structure which is non trivial and strict.

Let A be a non trivial set, let o_1, o_2 be binary operations on A, let n_1 be an element of A, and let n_2 be an element of $A \setminus \{n_1\}$. The functor field(A, o_1, o_2, n_1, n_2) yields a non trivial strict double loop structure and is defined by the conditions (Def. 14).

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[3] The definition (Def. 4) has been removed.

[4] The definitions (Def. 9)–(Def. 11) have been removed.

[5] The propositions (15)–(31) have been removed.

(Def. 14)(i) $A =$ the carrier of field(A, o_1, o_2, n_1, n_2),

(ii) $o_1 =$ the addition of field(A, o_1, o_2, n_1, n_2),

(iii) $o_2 =$ the multiplication of field(A, o_1, o_2, n_1, n_2),

(iv) $n_1 =$ the zero of field(A, o_1, o_2, n_1, n_2), and

(v) $n_2 =$ the unity of field(A, o_1, o_2, n_1, n_2).

Let X be a non trivial set, let F be a binary operation on X, and let x be an element of X. We say that F is binary operation preserving x if and only if:

(Def. 15) $X \setminus \{x\}$ is a set closed w.r.t. F and $F \restriction [: X \setminus \{x\}, X \setminus \{x\} :]$ is a binary operation on $X \setminus \{x\}$.

The following proposition is true

(39)[6] Let X be a set and A be a subset of X. Then there exists a binary operation F on X such that for every set x such that $x \in [: A, A :]$ holds $F(x) \in A$.

Let X be a set and let A be a subset of X. A binary operation on X is said to be a binary operation of X preserving A if:

(Def. 16) For every set x such that $x \in [: A, A :]$ holds it$(x) \in A$.

Next we state the proposition

(41)[7] Let X be a set, A be a subset of X, and F be a binary operation of X preserving A. Then $F \restriction [: A, A :]$ is a binary operation on A.

Let X be a set, let A be a subset of X, and let F be a binary operation of X preserving A. The functor $F \restriction A$ yielding a binary operation on A is defined by:

(Def. 17) $F \restriction A = F \restriction [: A, A :]$.

We now state the proposition

(43)[8] Let A be a non trivial set and x be an element of A. Then there exists a binary operation F on A such that for every set y if $y \in [: A \setminus \{x\}, A \setminus \{x\} :]$, then $F(y) \in A \setminus \{x\}$.

Let A be a non trivial set and let x be an element of A. A binary operation on A is said to be a binary operation of A preserving $A \setminus \{ x \}$ if:

(Def. 18) For every set y such that $y \in [: A \setminus \{x\}, A \setminus \{x\} :]$ holds it$(y) \in A \setminus \{x\}$.

Next we state the proposition

(45)[9] Let A be a non trivial set, x be an element of A, and F be a binary operation of A preserving $A \setminus \{ x \}$. Then $F \restriction [: A \setminus \{x\}, A \setminus \{x\} :]$ is a binary operation on $A \setminus \{x\}$.

Let A be a non trivial set, let x be an element of A, and let F be a binary operation of A preserving $A \setminus \{ x \}$. The functor $F \restriction_x A$ yields a binary operation on $A \setminus \{x\}$ and is defined as follows:

(Def. 19) $F \restriction_x A = F \restriction [: A \setminus \{x\}, A \setminus \{x\} :]$.

Let I_1 be a 1-sorted structure. Let us observe that I_1 is trivial if and only if:

(Def. 20) For all elements x, y of I_1 holds $x = y$.

[6] The propositions (36)–(38) have been removed.
[7] The proposition (40) has been removed.
[8] The proposition (42) has been removed.
[9] The proposition (44) has been removed.

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