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Projective Planes

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Summary. The line of points a, b, denoted by $a \cdot b$ and the point of lines A, B denoted by $A \cdot B$ are defined. A few basic theorems related to these notions are proved. An inspiration for such approach comes from so called Leibniz program. Let us recall that the Leibniz program is a program of algebraization of geometry using purely geometric notions. Leibniz formulated his program in opposition to algebraization method developed by Descartes.

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The articles [1], [5], [3], [4], and [2] provide the notation and terminology for this paper.

1. PROJECTIVE SPACES

We adopt the following convention: G is a projective incidence structure, a, a_1, a_2, b, b_1, b_2, c, d, p, q, r are points of G, and A, B, M, N, P, Q, R are lines of G.

Let us consider G, a, b, P. The predicate $a, b \nmid P$ is defined as follows:

(Def. 1) $a \nmid P$ and $b \nmid P$.

Let us consider G, a, P, Q. We say that a lies on P, Q if and only if:

(Def. 2) a lies on P and a lies on Q.

Let us consider G, a, P, Q, R. We say that a lies on P, Q, R if and only if:

(Def. 3) a lies on P and a lies on Q and a lies on R.

One can prove the following proposition

(1)(i) If a, b lie on P, then b, a lie on P,

(ii) if a, b, c lie on P, then a, c, b lie on P and b, a, c lie on P and b, c, a lie on P and c, a, b lie on P and c, b, a lie on P,

(iii) if a lies on P, Q, then a lies on Q, P, and

(iv) if a lies on P, Q, R, then a lies on P, R, Q and a lies on Q, P, R and a lies on Q, R, P and a lies on R, P, Q and a lies on R, Q, P.

Let I_1 be a projective incidence structure. We say that I_1 is configuration if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let p, q be points of I_1 and P, Q be lines of I_1. Suppose p lies on P and q lies on P and p lies on Q and q lies on Q. Then $p = q$ or $P = Q$.

Next we state three propositions:

(2) G is configuration iff for all p, q, P, Q such that p, q lie on P and p, q lie on Q holds $p = q$ or $P = Q$.

(3) G is configuration if and only if for all p, q, P, Q such that p lies on P, Q and q lies on P, Q holds $p = q$ or $P = Q$.

(4) The following statements are equivalent

(i) G is a projective space defined in terms of incidence,

(ii) G is configuration and for all p, q there exists P such that p, q lie on P and there exist p, P such that $p \nmid P$ and for every P there exist a, b, c such that a, b, c are mutually different and a, b, c lie on P and for all a, b, c, d, p, M, N, P, Q such that a, b, p lie on M and c, d, p lie on N and a, c lie on P and b, d lie on Q and $p \nmid P$ and $p \nmid Q$ and $M \neq N$ there exists q such that q lies on P, Q.

An incidence projective plane is a 2-dimensional projective space defined in terms of incidence. Let us consider G, a, b, c. We say that a, b and c are collinear if and only if:

(Def. 5) There exists P such that a, b, c lie on P.

We introduce a, b, c form a triangle as an antonym of a, b and c are collinear.
The following two propositions are true:

(5) a, b and c are collinear iff there exists P such that a lies on P and b lies on P and c lies on P.

(6) a, b, c form a triangle iff for every P holds $a \nmid P$ or $b \nmid P$ or $c \nmid P$.

Let us consider G, a, b, c, d. We say that a, b, c, d form a quadrangle if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) a, b, c form a triangle,

(ii) b, c, d form a triangle,

(iii) c, d, a form a triangle, and

(iv) d, a, b form a triangle.

Next we state several propositions:

(7) If G is a projective space defined in terms of incidence, then there exist A, B such that $A \neq B$.

(8) Suppose G is a projective space defined in terms of incidence and a lies on A. Then there exist b, c such that b, c lie on A and a, b, c are mutually different.

(9) Suppose G is a projective space defined in terms of incidence and a lies on A and $A \neq B$. Then there exists b such that b lies on A and $b \nmid B$ and $a \neq b$.

(10) If G is configuration and a_1, a_2 lie on A and $a_1 \neq a_2$ and $b \nmid A$, then a_1, a_2, b form a triangle.

(11) Suppose a, b and c are collinear. Then

(i) a, c and b are collinear,

(ii) b, a and c are collinear,

(iii) b, c and a are collinear,

(iv) c, a and b are collinear, and

(v) c, b and a are collinear.

(12) Suppose a, b, c form a triangle. Then

(i) a, c, b form a triangle,

(ii) b, a, c form a triangle,

(iii) b, c, a form a triangle,

(iv) c, a, b form a triangle, and

(v) c, b, a form a triangle.

(13) Suppose a, b, c, d form a quadrangle. Then a, c, b, d form a quadrangle and b, a, c, d form a quadrangle and b, c, a, d form a quadrangle and c, a, b, d form a quadrangle and c, b, a, d form a quadrangle and a, b, d, c form a quadrangle and a, c, d, b form a quadrangle and b, a, d, c form a quadrangle and b, c, d, a form a quadrangle and c, a, d, b form a quadrangle and c, b, d, a form a quadrangle and a, d, b, c form a quadrangle and a, d, c, b form a quadrangle and b, d, a, c form a quadrangle and b, d, c, a form a quadrangle and c, d, a, b form a quadrangle and c, d, b, a form a quadrangle and d, a, b, c form a quadrangle and d, a, c, b form a quadrangle and d, b, a, c form a quadrangle and d, b, c, a form a quadrangle and d, c, a, b form a quadrangle and d, c, b, a form a quadrangle.

(14) Suppose G is configuration and a_1, a_2 lie on A and b_1, b_2 lie on B and $a_1, a_2 \nmid B$ and $b_1, b_2 \nmid A$ and $a_1 \neq a_2$ and $b_1 \neq b_2$. Then a_1, a_2, b_1, b_2 form a quadrangle.

(15) Suppose G is a projective space defined in terms of incidence. Then there exist a, b, c, d such that a, b, c, d form a quadrangle.

Let G be a projective space defined in terms of incidence. An element of $[$:the points of G, the points of G, the points of G, the points of G:$]$ is said to be a quadrangle of G if:

(Def. 7) $\mathrm{it}_{\mathbf{1}}$, $\mathrm{it}_{\mathbf{2}}$, $\mathrm{it}_{\mathbf{3}}$, $\mathrm{it}_{\mathbf{4}}$ form a quadrangle.

Let G be a projective space defined in terms of incidence and let a, b be points of G. Let us assume that $a \neq b$. The functor $a \cdot b$ yielding a line of G is defined by:

(Def. 8) a, b lie on $a \cdot b$.

Next we state the proposition

(16) Let G be a projective space defined in terms of incidence, a, b be points of G, and L be a line of G. Suppose $a \neq b$. Then a lies on $a \cdot b$ and b lies on $a \cdot b$ and $a \cdot b = b \cdot a$ and if a lies on L and b lies on L, then $L = a \cdot b$.

2. Projective Planes

One can prove the following propositions:

(17) Suppose there exist a, b, c such that a, b, c form a triangle and for all p, q there exists M such that p, q lie on M. Then there exist p, P such that $p \nmid P$.

(18) If there exist a, b, c, d such that a, b, c, d form a quadrangle, then there exist a, b, c such that a, b, c form a triangle.

(19) If a, b, c form a triangle and a, b lie on P and a, c lie on Q, then $P \neq Q$.

(20) Suppose a, b, c, d form a quadrangle and a, b lie on P and a, c lie on Q and a, d lie on R. Then P, Q, R are mutually different.

(21) Suppose that G is configuration and a lies on P, Q, R and P, Q, R are mutually different and $a \nmid A$ and p lies on A, P and q lies on A, Q and r lies on A, R. Then p, q, r are mutually different.

(22) Suppose that

(i) G is configuration,

(ii) for all p, q there exists M such that p, q lie on M,

(iii) for all P, Q there exists a such that a lies on P, Q, and

(iv) there exist a, b, c, d such that a, b, c, d form a quadrangle.

Let given P. Then there exist a, b, c such that a, b, c are mutually different and a, b, c lie on P.

(23) G is an incidence projective plane if and only if the following conditions are satisfied:

(i) G is configuration,

(ii) for all p, q there exists M such that p, q lie on M,

(iii) for all P, Q there exists a such that a lies on P, Q, and

(iv) there exist a, b, c, d such that a, b, c, d form a quadrangle.

We use the following convention: G is an incidence projective plane, a, q are points of G, and A, B are lines of G.

Let us consider G, A, B. Let us assume that $A \neq B$. The functor $A \cdot B$ yielding a point of G is defined as follows:

(Def. 9) $A \cdot B$ lies on A, B.

We now state two propositions:

(24) Suppose $A \neq B$. Then $A \cdot B$ lies on A and $A \cdot B$ lies on B and $A \cdot B = B \cdot A$ and if a lies on A and a lies on B, then $a = A \cdot B$.

(25) If $A \neq B$ and a lies on A and $q \nmid A$ and $a \neq A \cdot B$, then $q \cdot a \cdot B$ lies on B and $q \cdot a \cdot B \nmid A$.

3. Some Useful Propositions

We adopt the following rules: G denotes a projective space defined in terms of incidence and a, b, c, d denote points of G.

Next we state two propositions:

(26) If a, b, c form a triangle, then a, b, c are mutually different.

(27) If a, b, c, d form a quadrangle, then a, b, c, d are mutually different.

In the sequel G denotes an incidence projective plane.

Next we state a number of propositions:

(28) For all points a, b, c, d of G such that $a \cdot c = b \cdot d$ holds $a = c$ or $b = d$ or $c = d$ or $a \cdot c = c \cdot d$.

(29) For all points a, b, c, d of G such that $a \cdot c = b \cdot d$ holds $a = c$ or $b = d$ or $c = d$ or a lies on $c \cdot d$.

(30) Let G be an incidence projective plane, e, m, m' be points of G, and I be a line of G. Suppose m lies on I and m' lies on I and $m \neq m'$ and $e \nmid I$. Then $m \cdot e \neq m' \cdot e$ and $e \cdot m \neq e \cdot m'$.

(31) Let G be an incidence projective plane, e be a point of G, and I, L_1, L_2 be lines of G. Suppose e lies on L_1 and e lies on L_2 and $L_1 \neq L_2$ and $e \nmid I$. Then $I \cdot L_1 \neq I \cdot L_2$ and $L_1 \cdot I \neq L_2 \cdot I$.

(32) Let G be a projective space defined in terms of incidence and a, b, q, q_1 be points of G. Suppose q lies on $a \cdot b$ and q lies on $a \cdot q_1$ and $q \neq a$ and $q_1 \neq a$ and $a \neq b$. Then q_1 lies on $a \cdot b$.

(33) Let G be a projective space defined in terms of incidence and a, b, c be points of G. If c lies on $a \cdot b$ and $a \neq c$, then b lies on $a \cdot c$.

(34) Let G be an incidence projective plane, q_1, q_2, r_1, r_2 be points of G, and H be a line of G. Suppose $r_1 \neq r_2$ and r_1 lies on H and r_2 lies on H and $q_1 \nmid H$ and $q_2 \nmid H$. Then $q_1 \cdot r_1 \neq q_2 \cdot r_2$.

(35) For all points a, b, c of G such that a lies on $b \cdot c$ holds $a = c$ or $b = c$ or b lies on $c \cdot a$.

(36) For all points a, b, c of G such that a lies on $b \cdot c$ holds $b = a$ or $b = c$ or c lies on $b \cdot a$.

(37) Let e, x_1, x_2, p_1, p_2 be points of G and H, I be lines of G. Suppose that x_1 lies on I and x_2 lies on I and e lies on H and $e \nmid I$ and $x_1 \neq x_2$ and $p_1 \neq e$ and $p_2 \neq e$ and p_1 lies on $e \cdot x_1$ and p_2 lies on $e \cdot x_2$. Then there exists a point r of G such that r lies on $p_1 \cdot p_2$ and r lies on H and $r \neq e$.

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