JOURNAL OF FORMALIZED MATHEMATICS
Volume 3, Released 1991, Published 2003
Inst. of Computer Science, Univ. of Białystok

Preliminaries to the Lambek Calculus

Wojciech Zielonka
Adam Mickiewicz University
Poznań

Summary. Some preliminary facts concerning completeness and decidability problems for the Lambek Calculus [14] are proved as well as some theses and derived rules of the calculus itself.

MML Identifier: PRELAMB.

WWW: http://mizar.org/JFM/Vol3/prelamb.html

The articles [16], [9], [19], [18], [13], [1], [6], [21], [5], [10], [12], [17], [20], [7], [8], [15], [11], [2], [3], and [4] provide the notation and terminology for this paper.

We consider structures of the type algebra as extensions of 1-sorted structure as systems

⟨ a carrier, a left quotient, a right quotient, an inner product ⟩,

where the carrier is a set and the left quotient, the right quotient, and the inner product are binary operations on the carrier.

One can check that there exists a structure of the type algebra which is non empty and strict.

Let s be a non empty structure of the type algebra. A type of s is an element of s.

We follow the rules: s denotes a non empty structure of the type algebra, T, X, Y denote finite sequences of elements of the carrier of s, and x, y, z denote types of s.

Let us consider s, x, y. The functor $x \setminus y$ yields a type of s and is defined by:

(Def. 1) $x \setminus y = (\text{the left quotient of } s)(x, y)$.

The functor x/y yields a type of s and is defined as follows:

(Def. 2) $x/y = (\text{the right quotient of } s)(x, y)$.

The functor $x \cdot y$ yielding a type of s is defined as follows:

(Def. 3) $x \cdot y = (\text{the inner product of } s)(x, y)$.

Let T_1 be a finite tree and let v be an element of T_1. One can check that $\mathrm{succ}\, v$ is finite.

Let T_1 be a finite tree and let v be an element of T_1. The branch degree of v is defined as follows:

(Def. 4) The branch degree of $v = \mathrm{card}\, \mathrm{succ}\, v$.

Let us note that there exists a decorated tree which is finite.

Let D be a non empty set. One can verify that there exists a tree decorated with elements of D which is finite.

Let us consider s. A preproof of s is a finite tree decorated with elements of $[: [: (\text{the carrier of } s)^*, \text{the carrier of } s :], \mathbb{N} :]$.

In the sequel T_1 is a preproof of s.

Let R be a finite binary relation. One can verify that $\mathrm{dom}\, R$ is finite.

Let us consider s, T_1 and let v be an element of $\mathrm{dom}\, T_1$. We say that v is correct if and only if:

(Def. 5)(i) The branch degree of $v = 0$ and there exists x such that $T_1(v)_{\mathbf{1}} = \langle\langle x\rangle, x\rangle$ if $T_1(v)_{\mathbf{2}} = 0$,

(ii) the branch degree of $v = 1$ and there exist T, x, y such that $T_1(v)_{\mathbf{1}} = \langle T, x/y\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T \frown \langle y\rangle, x\rangle$ if $T_1(v)_{\mathbf{2}} = 1$,

(iii) the branch degree of $v = 1$ and there exist T, x, y such that $T_1(v)_{\mathbf{1}} = \langle T, y \setminus x\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle\langle y\rangle \frown T, x\rangle$ if $T_1(v)_{\mathbf{2}} = 2$,

(iv) the branch degree of $v = 2$ and there exist T, X, Y, x, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown \langle x/y\rangle \frown T \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 3$,

(v) the branch degree of $v = 2$ and there exist T, X, Y, x, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown T \frown \langle y \setminus x\rangle \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 4$,

(vi) the branch degree of $v = 1$ and there exist X, x, y, Y such that $T_1(v)_{\mathbf{1}} = \langle X \frown \langle x \cdot y\rangle \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown \langle y\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 5$,

(vii) the branch degree of $v = 2$ and there exist X, Y, x, y such that $T_1(v)_{\mathbf{1}} = \langle X \frown Y, x \cdot y\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle X, x\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle Y, y\rangle$ if $T_1(v)_{\mathbf{2}} = 6$,

(viii) the branch degree of $v = 2$ and there exist T, X, Y, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown T \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle y\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 7$,

(ix) *contradiction*;[1] otherwise.

Let us consider s and let I_1 be a type of s. We say that I_1 is left if and only if:

(Def. 6) There exist x, y such that $I_1 = x \setminus y$.

We say that I_1 is right if and only if:

(Def. 7) There exist x, y such that $I_1 = x/y$.

We say that I_1 is middle if and only if:

(Def. 8) There exist x, y such that $I_1 = x \cdot y$.

Let us consider s and let I_1 be a type of s. We say that I_1 is primitive if and only if:

(Def. 9) I_1 is left, right, and middle.

Let us consider s, let T_1 be a finite tree decorated with elements of the carrier of s, and let v be an element of $\operatorname{dom} T_1$. Then $T_1(v)$ is a type of s.

Let us consider s, let T_1 be a finite tree decorated with elements of the carrier of s, and let us consider x. We say that T_1 represents x if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) $\operatorname{dom} T_1$ is finite, and

(ii) for every element v of $\operatorname{dom} T_1$ holds the branch degree of $v = 0$ or the branch degree of $v = 2$ but if the branch degree of $v = 0$, then $T_1(v)$ is primitive but if the branch degree of $v = 2$, then there exist y, z such that $T_1(v) = y/z$ or $T_1(v) = y \setminus z$ or $T_1(v) = y \cdot z$ but $T_1(v \frown \langle 0\rangle) = y$ but $T_1(v \frown \langle 1\rangle) = z$.

We introduce T_1 does not represent x as an antonym of T_1 represents x.

Let I_1 be a non empty structure of the type algebra. We say that I_1 is free if and only if the conditions (Def. 11) are satisfied.

(Def. 11)(i) There exists no type of I_1 which is left, right, left, middle, right, and middle, and

(ii) for every type x of I_1 there exists a finite tree T_1 decorated with elements of the carrier of I_1 such that for every finite tree T_2 decorated with elements of the carrier of I_1 holds T_2 represents x iff $T_1 = T_2$.

Let us consider s, x. Let us assume that s is free. The representation of x yielding a finite tree decorated with elements of the carrier of s is defined by the condition (Def. 12).

[1] This definition is absolutely permissive, i.e. we assume a *contradiction*, but we are interested only in the type of the functor 'choose'.

(Def. 12) Let T_1 be a finite tree decorated with elements of the carrier of s. Then T_1 represents x if and only if the representation of $x = T_1$.

Let us consider s, let f be a finite sequence of elements of the carrier of s, and let t be a type of s. Then $\langle f, t\rangle$ is an element of $[:(\text{the carrier of } s)^*, \text{the carrier of } s:]$.

Let us consider s. A preproof of s is called a proof of s if:

(Def. 13) dom it is finite and every element of dom it is correct.

In the sequel p denotes a proof of s and v denotes an element of dom p.

One can prove the following propositions:

(1) If the branch degree of $v = 1$, then $v \frown \langle 0\rangle \in \operatorname{dom} p$.

(2) If the branch degree of $v = 2$, then $v \frown \langle 0\rangle \in \operatorname{dom} p$ and $v \frown \langle 1\rangle \in \operatorname{dom} p$.

(3) If $p(v)_{\mathbf{2}} = 0$, then there exists x such that $p(v)_{\mathbf{1}} = \langle\langle x\rangle, x\rangle$.

(4) If $p(v)_{\mathbf{2}} = 1$, then there exists an element w of dom p and there exist T, x, y such that $w = v \frown \langle 0\rangle$ and $p(v)_{\mathbf{1}} = \langle T, x/y\rangle$ and $p(w)_{\mathbf{1}} = \langle T \frown \langle y\rangle, x\rangle$.

(5) If $p(v)_{\mathbf{2}} = 2$, then there exists an element w of dom p and there exist T, x, y such that $w = v \frown \langle 0\rangle$ and $p(v)_{\mathbf{1}} = \langle T, y \backslash x\rangle$ and $p(w)_{\mathbf{1}} = \langle\langle y\rangle \frown T, x\rangle$.

(6) Suppose $p(v)_{\mathbf{2}} = 3$. Then there exist elements w, u of dom p and there exist T, X, Y, x, y, z such that $w = v \frown \langle 0\rangle$ and $u = v \frown \langle 1\rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown \langle x/y\rangle \frown T \frown Y, z\rangle$ and $p(w)_{\mathbf{1}} = \langle T, y\rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$.

(7) Suppose $p(v)_{\mathbf{2}} = 4$. Then there exist elements w, u of dom p and there exist T, X, Y, x, y, z such that $w = v \frown \langle 0\rangle$ and $u = v \frown \langle 1\rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown T \frown \langle y \backslash x\rangle \frown Y, z\rangle$ and $p(w)_{\mathbf{1}} = \langle T, y\rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$.

(8) Suppose $p(v)_{\mathbf{2}} = 5$. Then there exists an element w of dom p and there exist X, x, y, Y such that $w = v \frown \langle 0\rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown \langle x \cdot y\rangle \frown Y, z\rangle$ and $p(w)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown \langle y\rangle \frown Y, z\rangle$.

(9) Suppose $p(v)_{\mathbf{2}} = 6$. Then there exist elements w, u of dom p and there exist X, Y, x, y such that $w = v \frown \langle 0\rangle$ and $u = v \frown \langle 1\rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown Y, x \cdot y\rangle$ and $p(w)_{\mathbf{1}} = \langle X, x\rangle$ and $p(u)_{\mathbf{1}} = \langle Y, y\rangle$.

(10) Suppose $p(v)_{\mathbf{2}} = 7$. Then there exist elements w, u of dom p and there exist T, X, Y, y, z such that $w = v \frown \langle 0\rangle$ and $u = v \frown \langle 1\rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown T \frown Y, z\rangle$ and $p(w)_{\mathbf{1}} = \langle T, y\rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle y\rangle \frown Y, z\rangle$.

(11) $p(v)_{\mathbf{2}} = 0$ or $p(v)_{\mathbf{2}} = 1$ or $p(v)_{\mathbf{2}} = 2$ or $p(v)_{\mathbf{2}} = 3$ or $p(v)_{\mathbf{2}} = 4$ or $p(v)_{\mathbf{2}} = 5$ or $p(v)_{\mathbf{2}} = 6$ or $p(v)_{\mathbf{2}} = 7$.

Let us consider s and let I_1 be a preproof of s. We say that I_1 is cut-free if and only if:

(Def. 14) For every element v of dom I_1 holds $I_1(v)_{\mathbf{2}} \neq 7$.

Let us consider s. The size w.r.t. s yielding a function from the carrier of s into $\mathbb{N}$ is defined by:

(Def. 15) For every x holds (the size w.r.t. s)$(x) = \operatorname{card} \operatorname{dom}$(the representation of x).

Let D be a non empty set, let T be a finite sequence of elements of D, and let f be a function from D into $\mathbb{N}$. Then $f \cdot T$ is a finite sequence of elements of $\mathbb{R}$.

Let us consider s and let p be a proof of s. The cut degree of p yielding a natural number is defined by:

(Def. 16)(i) There exist T, X, Y, y, z such that $p(\emptyset)_{\mathbf{1}} = \langle X \frown T \frown Y, z\rangle$ and $p(\langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $p(\langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle y\rangle \frown Y, z\rangle$ and the cut degree of $p =$ (the size w.r.t. s)(y) + (the size w.r.t. s)$(z) + \sum$((the size w.r.t. s) $\cdot (X \frown T \frown Y)$) if $p(\emptyset)_{\mathbf{2}} = 7$,

(ii) the cut degree of $p = 0$, otherwise.

We use the following convention: A is a non empty set and a, a_1, a_2, b are elements of A^*.

Let us consider s, A. A function from the carrier of s into 2^{A^*} is said to be a model of s if it satisfies the condition (Def. 17).

(Def. 17) Let given x, y. Then

(i) $\mathrm{it}(x \cdot y) = \{a \frown b : a \in \mathrm{it}(x) \ \wedge \ b \in \mathrm{it}(y)\}$,

(ii) $\mathrm{it}(x/y) = \{a_1 : \bigwedge_b \ (b \in \mathrm{it}(y) \ \Rightarrow \ a_1 \frown b \in \mathrm{it}(x))\}$, and

(iii) $\mathrm{it}(y \backslash x) = \{a_2 : \bigwedge_b \ (b \in \mathrm{it}(y) \ \Rightarrow \ b \frown a_2 \in \mathrm{it}(x))\}$.

Let a, b be non empty sets. Observe that there exists a relation between a and b which is non empty.

We consider type structures as extensions of structure of the type algebra as systems

⟨ a carrier, a left quotient, a right quotient, an inner product, a derivability ⟩,

where the carrier is a set, the left quotient, the right quotient, and the inner product are binary operations on the carrier, and the derivability is a relation between the carrier* and the carrier.

Let us observe that there exists a type structure which is non empty and strict.

In the sequel s is a non empty type structure and x is a type of s.

Let us consider s, let f be a finite sequence of elements of the carrier of s, and let us consider x. The predicate $f \longrightarrow x$ is defined by:

(Def. 18) $\langle f, x \rangle \in$ the derivability of s.

Let I_1 be a non empty type structure. We say that I_1 is calculus of syntactic types-like if and only if the conditions (Def. 19) are satisfied.

(Def. 19) For every type x of I_1 holds $\langle x \rangle \longrightarrow x$ and for every finite sequence T of elements of the carrier of I_1 and for all types x, y of I_1 such that $T \frown \langle y \rangle \longrightarrow x$ holds $T \longrightarrow x/y$ and for every finite sequence T of elements of the carrier of I_1 and for all types x, y of I_1 such that $\langle y \rangle \frown T \longrightarrow x$ holds $T \longrightarrow y \backslash x$ and for all finite sequences T, X, Y of elements of the carrier of I_1 and for all types x, y, z of I_1 such that $T \longrightarrow y$ and $X \frown \langle x \rangle \frown Y \longrightarrow z$ holds $X \frown \langle x/y \rangle \frown T \frown Y \longrightarrow z$ and for all finite sequences T, X, Y of elements of the carrier of I_1 and for all types x, y, z of I_1 such that $T \longrightarrow y$ and $X \frown \langle x \rangle \frown Y \longrightarrow z$ holds $X \frown T \frown \langle y \backslash x \rangle \frown Y \longrightarrow z$ and for all finite sequences X, Y of elements of the carrier of I_1 and for all types x, y, z of I_1 such that $X \frown \langle x \rangle \frown \langle y \rangle \frown Y \longrightarrow z$ holds $X \frown \langle x \cdot y \rangle \frown Y \longrightarrow z$ and for all finite sequences X, Y of elements of the carrier of I_1 and for all types x, y of I_1 such that $X \longrightarrow x$ and $Y \longrightarrow y$ holds $X \frown Y \longrightarrow x \cdot y$.

Let us observe that there exists a non empty type structure which is calculus of syntactic types-like.

A calculus of syntactic types is a calculus of syntactic types-like non empty type structure.

In the sequel s denotes a calculus of syntactic types and x, y, z denote types of s.

Next we state a number of propositions:

(12) $\langle x/y \rangle \frown \langle y \rangle \longrightarrow x$ and $\langle y \rangle \frown \langle y \backslash x \rangle \longrightarrow x$.

(13) $\langle x \rangle \longrightarrow y/(x \backslash y)$ and $\langle x \rangle \longrightarrow y/x \backslash y$.

(14) $\langle x/y \rangle \longrightarrow x/z/(y/z)$.

(15) $\langle y \backslash x \rangle \longrightarrow z \backslash y \backslash (z \backslash x)$.

(16) If $\langle x \rangle \longrightarrow y$, then $\langle x/z \rangle \longrightarrow y/z$ and $\langle z \backslash x \rangle \longrightarrow z \backslash y$.

(17) If $\langle x \rangle \longrightarrow y$, then $\langle z/y \rangle \longrightarrow z/x$ and $\langle y \backslash z \rangle \longrightarrow x \backslash z$.

(18) $\langle y/(y/x \backslash y) \rangle \longrightarrow y/x$.

(19) If $\langle x \rangle \longrightarrow y$, then $\varepsilon_{(\text{the carrier of } s)} \longrightarrow y/x$ and $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x \backslash y$.

(20) $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x/x$ and $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x \backslash x$.

(21) $\varepsilon_{(\text{the carrier of } s)} \longrightarrow y/(x \backslash y)/x$ and $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x \backslash (y/x \backslash y)$.

(22) $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x/z/(y/z)/(x/y)$ and $\varepsilon_{(\text{the carrier of } s)} \longrightarrow y \backslash x \backslash (z \backslash y \backslash (z \backslash x))$.

(23) If $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x$, then $\varepsilon_{(\text{the carrier of } s)} \longrightarrow y/(y/x)$ and $\varepsilon_{(\text{the carrier of } s)} \longrightarrow x \backslash y \backslash y$.

(24) $\langle x/(y/y)\rangle \longrightarrow x$.

Let us consider s, x, y. The predicate $x \longleftrightarrow y$ is defined as follows:

(Def. 20) $\langle x\rangle \longrightarrow y$ and $\langle y\rangle \longrightarrow x$.

The following propositions are true:

(25) $x \longleftrightarrow x$.

(26) $x/y \longleftrightarrow x/(x/y \backslash x)$.

(27) $x/(z \cdot y) \longleftrightarrow x/y/z$.

(28) $\langle x \cdot (y/z)\rangle \longrightarrow (x \cdot y)/z$.

(29) $\langle x\rangle \longrightarrow (x \cdot y)/y$ and $\langle x\rangle \longrightarrow y \backslash y \cdot x$.

(30) $(x \cdot y) \cdot z \longleftrightarrow x \cdot (y \cdot z)$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[3] Grzegorz Bancerek. Introduction to trees. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/trees_1.html.

[4] Grzegorz Bancerek. König's Lemma. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/trees_2.html.

[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[6] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[7] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[8] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[9] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[10] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[11] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rvsum_1.html.

[12] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[14] Joachim Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, (65):154–170, 1958.

[15] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[17] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/mcart_1.html.

[18] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[19] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[20] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[21] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

Received February 13, 1991

Published January 2, 2004