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Product of Family of Universal Algebras

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Summary. The product of two algebras, trivial algebra determined by an empty set and product of a family of algebras are defined. Some basic properties are shown.

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The articles [17], [11], [21], [1], [2], [20], [22], [8], [5], [9], [15], [18], [16], [10], [12], [13], [3], [4], [6], [14], [7], and [19] provide the notation and terminology for this paper.

1. Product of Two Algebras

One can prove the following proposition

(1) For all non empty sets D_1, D_2 and for all natural numbers n, m such that ${D_1}^n = {D_2}^m$ holds $n = m$.

For simplicity, we follow the rules: U_1, U_2 denote universal algebras, n, m denote natural numbers, x, y denote sets, A, B denote non empty sets, and h_1 denotes a finite sequence of elements of $[:A, B:]$.

Let us consider A, B and let us consider h_1. Then $\mathrm{pr1}(h_1)$ is a finite sequence of elements of A and it can be characterized by the condition:

(Def. 1) $\mathrm{len}\,\mathrm{pr1}(h_1) = \mathrm{len}\,h_1$ and for every n such that $n \in \mathrm{dom}\,\mathrm{pr1}(h_1)$ holds $\mathrm{pr1}(h_1)(n) = h_1(n)_{\mathbf{1}}$.

Then $\mathrm{pr2}(h_1)$ is a finite sequence of elements of B and it can be characterized by the condition:

(Def. 2) $\mathrm{len}\,\mathrm{pr2}(h_1) = \mathrm{len}\,h_1$ and for every n such that $n \in \mathrm{dom}\,\mathrm{pr2}(h_1)$ holds $\mathrm{pr2}(h_1)(n) = h_1(n)_{\mathbf{2}}$.

Let us consider A, B, let f_1 be a homogeneous quasi total non empty partial function from A^* to A, and let f_2 be a homogeneous quasi total non empty partial function from B^* to B. Let us assume that $\mathrm{arity}\,f_1 = \mathrm{arity}\,f_2$. The functor $]\!]f_1, f_2[\![$ yields a homogeneous quasi total non empty partial function from $[:A, B:]^*$ to $[:A, B:]$ and is defined by the conditions (Def. 3).

(Def. 3)(i) $\mathrm{dom}\,]\!]f_1, f_2[\![= [:A, B:]^{\mathrm{arity}\,f_1}$, and

(ii) for every finite sequence h of elements of $[:A, B:]$ such that $h \in \mathrm{dom}\,]\!]f_1, f_2[\![$ holds $]\!]f_1, f_2[\![(h) = \langle f_1(\mathrm{pr1}(h)), f_2(\mathrm{pr2}(h))\rangle$.

In the sequel h_1 denotes a homogeneous quasi total non empty partial function from (the carrier of $U_1)^*$ to the carrier of U_1 and h_2 denotes a homogeneous quasi total non empty partial function from (the carrier of $U_2)^*$ to the carrier of U_2.

Let us consider U_1, U_2. Let us assume that U_1 and U_2 are similar. The functor $\mathrm{Opers}(U_1, U_2)$ yields a finite sequence of operational functions of $[:$the carrier of U_1, the carrier of $U_2:]$ and is defined by the conditions (Def. 4).

(Def. 4)(i) $\operatorname{len}\operatorname{Opers}(U_1,U_2) = \operatorname{len}$(the characteristic of U_1), and

(ii) for every n such that $n \in \operatorname{dom}\operatorname{Opers}(U_1,U_2)$ and for all h_1, h_2 such that $h_1 =$ (the characteristic of U_1)(n) and $h_2 =$ (the characteristic of U_2)(n) holds $(\operatorname{Opers}(U_1,U_2))(n) =]\!]h_1,h_2[\![$.

One can prove the following proposition

(2) If U_1 and U_2 are similar, then $\langle[\!:$the carrier of U_1, the carrier of $U_2:\!], \operatorname{Opers}(U_1,U_2)\rangle$ is a strict universal algebra.

Let us consider U_1, U_2. Let us assume that U_1 and U_2 are similar. The functor $[\!:U_1, U_2:\!]$ yielding a strict universal algebra is defined as follows:

(Def. 5) $[\!:U_1, U_2:\!] = \langle[\!:$the carrier of U_1, the carrier of $U_2:\!], \operatorname{Opers}(U_1,U_2)\rangle$.

Let A, B be non empty sets. The functor $\operatorname{Inv}(A,B)$ yielding a function from $[\!:A, B:\!]$ into $[\!:B, A:\!]$ is defined as follows:

(Def. 6) For every element a of $[\!:A, B:\!]$ holds $(\operatorname{Inv}(A,B))(a) = \langle a_{\mathbf{2}}, a_{\mathbf{1}}\rangle$.

Next we state several propositions:

(3) For all non empty sets A, B holds $\operatorname{rng}\operatorname{Inv}(A,B) = [\!:B, A:\!]$.

(4) For all non empty sets A, B holds $\operatorname{Inv}(A,B)$ is one-to-one.

(5) Suppose U_1 and U_2 are similar. Then Inv(the carrier of U_1, the carrier of U_2) is a function from the carrier of $[\!:U_1, U_2:\!]$ into the carrier of $[\!:U_2, U_1:\!]$.

(6) Suppose U_1 and U_2 are similar. Let o_1 be an operation of U_1, o_2 be an operation of U_2, o be an operation of $[\!:U_1, U_2:\!]$, and n be a natural number. Suppose that

(i) $o_1 =$ (the characteristic of U_1)(n),

(ii) $o_2 =$ (the characteristic of U_2)(n),

(iii) $o =$ (the characteristic of $[\!:U_1, U_2:\!]$)(n), and

(iv) $n \in \operatorname{dom}$(the characteristic of U_1).

Then $\operatorname{arity} o = \operatorname{arity} o_1$ and $\operatorname{arity} o = \operatorname{arity} o_2$ and $o =]\!]o_1,o_2[\![$.

(7) If U_1 and U_2 are similar, then $[\!:U_1, U_2:\!]$ and U_1 are similar.

(8) Let U_1, U_2, U_3, U_4 be universal algebras. Suppose U_1 is a subalgebra of U_2 and U_3 is a subalgebra of U_4 and U_2 and U_4 are similar. Then $[\!:U_1, U_3:\!]$ is a subalgebra of $[\!:U_2, U_4:\!]$.

2. Trivial Algebra

Let k be a natural number. The functor $\operatorname{TrivOp}(k)$ yielding a partial function from $\{\emptyset\}^*$ to $\{\emptyset\}$ is defined by:

(Def. 7) $\operatorname{dom}\operatorname{TrivOp}(k) = \{k \mapsto \emptyset\}$ and $\operatorname{rng}\operatorname{TrivOp}(k) = \{\emptyset\}$.

Let k be a natural number. Note that $\operatorname{TrivOp}(k)$ is homogeneous, quasi total, and non empty.
The following proposition is true

(9) For every natural number k holds $\operatorname{arity}\operatorname{TrivOp}(k) = k$.

Let f be a finite sequence of elements of $\mathbb{N}$. The functor $\operatorname{TrivOps}(f)$ yields a finite sequence of operational functions of $\{\emptyset\}$ and is defined as follows:

(Def. 8) $\operatorname{len}\operatorname{TrivOps}(f) = \operatorname{len} f$ and for every n such that $n \in \operatorname{dom}\operatorname{TrivOps}(f)$ and for every m such that $m = f(n)$ holds $(\operatorname{TrivOps}(f))(n) = \operatorname{TrivOp}(m)$.

One can prove the following two propositions:

(10) For every finite sequence f of elements of $\mathbb{N}$ holds $\mathrm{TrivOps}(f)$ is homogeneous, quasi total, and non-empty.

(11) For every finite sequence f of elements of $\mathbb{N}$ such that $f \neq \emptyset$ holds $\langle \{\emptyset\}, \mathrm{TrivOps}(f) \rangle$ is a strict universal algebra.

Let D be a non empty set. Observe that there exists a finite sequence of elements of D which is non empty and there exists an element of D^* which is non empty.

Let f be a non empty finite sequence of elements of $\mathbb{N}$. The trivial algebra of f yielding a strict universal algebra is defined as follows:

(Def. 9) The trivial algebra of $f = \langle \{\emptyset\}, \mathrm{TrivOps}(f) \rangle$.

3. Product of Universal Algebras

Let I_1 be a function. We say that I_1 is universal algebra yielding if and only if:

(Def. 10) For every x such that $x \in \mathrm{dom}\, I_1$ holds $I_1(x)$ is a universal algebra.

Let I_1 be a function. We say that I_1 is 1-sorted yielding if and only if:

(Def. 11) For every x such that $x \in \mathrm{dom}\, I_1$ holds $I_1(x)$ is a 1-sorted structure.

Let us mention that there exists a function which is universal algebra yielding.

Let us mention that every function which is universal algebra yielding is also 1-sorted yielding.

Let I be a set. Observe that there exists a many sorted set indexed by I which is 1-sorted yielding.

Let I_1 be a function. We say that I_1 is equal signature if and only if:

(Def. 12) For all x, y such that $x \in \mathrm{dom}\, I_1$ and $y \in \mathrm{dom}\, I_1$ and for all U_1, U_2 such that $U_1 = I_1(x)$ and $U_2 = I_1(y)$ holds $\mathrm{signature}\, U_1 = \mathrm{signature}\, U_2$.

Let J be a non empty set. One can verify that there exists a many sorted set indexed by J which is equal signature and universal algebra yielding.

Let J be a non empty set, let A be a 1-sorted yielding many sorted set indexed by J, and let j be an element of J. Then $A(j)$ is a 1-sorted structure.

Let J be a non empty set, let A be a universal algebra yielding many sorted set indexed by J, and let j be an element of J. Then $A(j)$ is a universal algebra.

Let J be a set and let A be a 1-sorted yielding many sorted set indexed by J. The support of A yielding a many sorted set indexed by J is defined by the condition (Def. 13).

(Def. 13) Let j be a set. Suppose $j \in J$. Then there exists a 1-sorted structure R such that $R = A(j)$ and (the support of $A)(j) =$ the carrier of R.

Let J be a non empty set and let A be a universal algebra yielding many sorted set indexed by J. One can check that the support of A is non-empty.

Let J be a non empty set and let A be an equal signature universal algebra yielding many sorted set indexed by J. The functor $\mathrm{ComSign}(A)$ yields a finite sequence of elements of $\mathbb{N}$ and is defined as follows:

(Def. 14) For every element j of J holds $\mathrm{ComSign}(A) = \mathrm{signature}\, A(j)$.

Let I_1 be a function. We say that I_1 is function yielding if and only if:

(Def. 15) For every x such that $x \in \mathrm{dom}\, I_1$ holds $I_1(x)$ is a function.

Let us observe that there exists a function which is function yielding.

Let I be a set. One can check that there exists a many sorted set indexed by I which is function yielding.

Let I be a set. A many sorted function indexed by I is a function yielding many sorted set indexed by I.

Let B be a function yielding function and let j be a set. One can verify that $B(j)$ is function-like and relation-like.

Let J be a non empty set, let B be a non-empty many sorted set indexed by J, and let j be an element of J. Observe that $B(j)$ is non empty.

Let F be a function yielding function and let f be a function. Note that $F \cdot f$ is function yielding.

Let J be a non empty set and let B be a non-empty many sorted set indexed by J. Note that $\prod B$ is non empty.

Let J be a non empty set and let B be a non-empty many sorted set indexed by J. A many sorted function indexed by J is said to be a many sorted operation of B if:

(Def. 16) For every element j of J holds $\mathrm{it}(j)$ is a homogeneous quasi total non empty partial function from $B(j)^*$ to $B(j)$.

Let J be a non empty set, let B be a non-empty many sorted set indexed by J, let O be a many sorted operation of B, and let j be an element of J. Then $O(j)$ is a homogeneous quasi total non empty partial function from $B(j)^*$ to $B(j)$.

Let I_1 be a function. We say that I_1 is equal arity if and only if the condition (Def. 17) is satisfied.

(Def. 17) Let x, y be sets. Suppose $x \in \mathrm{dom}\, I_1$ and $y \in \mathrm{dom}\, I_1$. Let f, g be functions. Suppose $I_1(x) = f$ and $I_1(y) = g$. Let n, m be natural numbers and X, Y be non empty sets. Suppose $\mathrm{dom}\, f = X^n$ and $\mathrm{dom}\, g = Y^m$. Let o_1 be a homogeneous quasi total non empty partial function from X^* to X and o_2 be a homogeneous quasi total non empty partial function from Y^* to Y. If $f = o_1$ and $g = o_2$, then $\mathrm{arity}\, o_1 = \mathrm{arity}\, o_2$.

Let J be a non empty set and let B be a non-empty many sorted set indexed by J. Note that there exists a many sorted operation of B which is equal arity.

The following proposition is true

(12) Let J be a non empty set, B be a non-empty many sorted set indexed by J, and O be a many sorted operation of B. Then O is equal arity if and only if for all elements i, j of J holds $\mathrm{arity}\, O(i) = \mathrm{arity}\, O(j)$.

Let F be a function yielding function and let f be a function. The functor $F \looparrowleft f$ yields a function and is defined as follows:

(Def. 18) $\mathrm{dom}(F \looparrowleft f) = \mathrm{dom}\, F$ and for every set x such that $x \in \mathrm{dom}\, F$ holds $(F \looparrowleft f)(x) = F(x)(f(x))$.

Let I be a set, let f be a many sorted function indexed by I, and let x be a many sorted set indexed by I. Then $f \looparrowleft x$ is a many sorted set indexed by I and it can be characterized by the condition:

(Def. 19) For every set i such that $i \in I$ and for every function g such that $g = f(i)$ holds $(f \looparrowleft x)(i) = g(x(i))$.

Let J be a non empty set, let B be a non-empty many sorted set indexed by J, and let p be a finite sequence of elements of $\prod B$. Then $\mathrm{uncurry}\, p$ is a many sorted set indexed by $[: \mathrm{dom}\, p, J :]$.

Let I, J be sets and let X be a many sorted set indexed by $[: I, J :]$. Then $\curvearrowleft X$ is a many sorted set indexed by $[: J, I :]$.

Let X be a set, let Y be a non empty set, and let f be a many sorted set indexed by $[: X, Y :]$. Then $\mathrm{curry}\, f$ is a many sorted set indexed by X.

Let J be a non empty set, let B be a non-empty many sorted set indexed by J, and let O be an equal arity many sorted operation of B. The functor $\mathrm{ComAr}(O)$ yields a natural number and is defined as follows:

(Def. 20) For every element j of J holds $\mathrm{ComAr}(O) = \mathrm{arity}\, O(j)$.

Let I be a set and let A be a many sorted set indexed by I. The functor ε_A yielding a many sorted set indexed by I is defined as follows:

(Def. 21) For every set i such that $i \in I$ holds $\varepsilon_A(i) = \emptyset_{A(i)}$.

Let J be a non empty set, let B be a non-empty many sorted set indexed by J, and let O be an equal arity many sorted operation of B. The functor $\rceil\rceil O \lceil\lceil$ yields a homogeneous quasi total non empty partial function from $(\prod B)^*$ to $\prod B$ and is defined by the conditions (Def. 22).

(Def. 22)(i) $\mathrm{dom}\rceil\rceil O\lceil\lceil = (\prod B)^{\mathrm{ComAr}(O)}$, and

(ii) for every element p of $(\prod B)^*$ such that $p \in \mathrm{dom}\rceil\rceil O\lceil\lceil$ holds if $\mathrm{dom}\, p = \emptyset$, then $\rceil\rceil O\lceil\lceil(p) = O \hookleftarrow (\varepsilon_B)$ and if $\mathrm{dom}\, p \neq \emptyset$, then for every non empty set Z and for every many sorted set w indexed by $[: J, Z :]$ such that $Z = \mathrm{dom}\, p$ and $w = \curvearrowleft \mathrm{uncurry}\, p$ holds $\rceil\rceil O\lceil\lceil(p) = O \hookleftarrow \mathrm{curry}\, w$.

Let J be a non empty set, let A be an equal signature universal algebra yielding many sorted set indexed by J, and let n be a natural number. Let us assume that $n \in \mathrm{dom}\,\mathrm{ComSign}(A)$. The functor $\mathrm{ProdOp}(A,n)$ yielding an equal arity many sorted operation of the support of A is defined by the condition (Def. 23).

(Def. 23) Let j be an element of J and o be an operation of $A(j)$. If (the characteristic of $A(j))(n) = o$, then $(\mathrm{ProdOp}(A,n))(j) = o$.

Let J be a non empty set and let A be an equal signature universal algebra yielding many sorted set indexed by J. The functor $\mathrm{ProdOpSeq}(A)$ yields a finite sequence of operational functions of $\prod$(the support of A) and is defined as follows:

(Def. 24) $\mathrm{len}\,\mathrm{ProdOpSeq}(A) = \mathrm{len}\,\mathrm{ComSign}(A)$ and for every n such that $n \in \mathrm{dom}\,\mathrm{ProdOpSeq}(A)$ holds $(\mathrm{ProdOpSeq}(A))(n) = \rceil\rceil \mathrm{ProdOp}(A,n)\lceil\lceil$.

Let J be a non empty set and let A be an equal signature universal algebra yielding many sorted set indexed by J. The functor $\mathrm{ProdUnivAlg}(A)$ yields a strict universal algebra and is defined by:

(Def. 25) $\mathrm{ProdUnivAlg}(A) = \langle \prod$(the support of A),$\mathrm{ProdOpSeq}(A)\rangle$.

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