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Fundamental Theorem of Algebra[1]

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The articles [26], [34], [3], [28], [10], [29], [2], [22], [24], [30], [19], [13], [5], [35], [6], [7], [27], [4], [33], [31], [18], [12], [11], [1], [8], [23], [21], [9], [32], [14], [15], [25], [20], [17], and [16] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all natural numbers n, m such that $n \neq 0$ and $m \neq 0$ holds $(n \cdot m - n - m) + 1 \geq 0$.

(2) For all real numbers x, y such that $y > 0$ holds $\frac{\min(x,y)}{\max(x,y)} \leq 1$.

(3) For all real numbers x, y such that for every real number c such that $c > 0$ and $c < 1$ holds $c \cdot x \geq y$ holds $y \leq 0$.

(4) Let p be a finite sequence of elements of $\mathbb{R}$. Suppose that for every natural number n such that $n \in \operatorname{dom} p$ holds $p(n) \geq 0$. Let i be a natural number. If $i \in \operatorname{dom} p$, then $\sum p \geq p(i)$.

(5) For all real numbers x, y holds $-(x + y i_{\mathbb{C}_F}) = -x + (-y) i_{\mathbb{C}_F}$.

(6) For all real numbers x_1, y_1, x_2, y_2 holds $(x_1 + y_1 i_{\mathbb{C}_F}) - (x_2 + y_2 i_{\mathbb{C}_F}) = (x_1 - x_2) + (y_1 - y_2) i_{\mathbb{C}_F}$.

In this article we present several logical schemes. The scheme *ExDHGrStrSeq* deals with a non empty groupoid $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

> There exists a sequence S of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

The scheme *ExDdoubleLoopStrSeq* deals with a non empty double loop structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

> There exists a sequence S of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

The following proposition is true

(8)[1] For every element z of $\mathbb{C}_F$ such that $z \neq 0_{\mathbb{C}_F}$ and for every natural number n holds $|\mathrm{power}_{\mathbb{C}_F}(z, n)| = |z|^n$.

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[1] The proposition (7) has been removed.

Let p be a finite sequence of elements of the carrier of $\mathbb{C}_{\rm F}$. The functor $|p|$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 1) $\operatorname{len}|p| = \operatorname{len} p$ and for every natural number n such that $n \in \operatorname{dom} p$ holds $|p|_n = |p_n|$.

We now state several propositions:

(9) $|\varepsilon_{(\text{the carrier of } \mathbb{C}_{\rm F})}| = \varepsilon_{\mathbb{R}}$.

(10) For every element x of $\mathbb{C}_{\rm F}$ holds $|\langle x \rangle| = \langle |x| \rangle$.

(11) For all elements x, y of $\mathbb{C}_{\rm F}$ holds $|\langle x, y \rangle| = \langle |x|, |y| \rangle$.

(12) For all elements x, y, z of $\mathbb{C}_{\rm F}$ holds $|\langle x, y, z \rangle| = \langle |x|, |y|, |z| \rangle$.

(13) For all finite sequences p, q of elements of the carrier of $\mathbb{C}_{\rm F}$ holds $|p \frown q| = |p| \frown |q|$.

(14) Let p be a finite sequence of elements of the carrier of $\mathbb{C}_{\rm F}$ and x be an element of $\mathbb{C}_{\rm F}$. Then $|p \frown \langle x \rangle| = |p| \frown \langle |x| \rangle$ and $|\langle x \rangle \frown p| = \langle |x| \rangle \frown |p|$.

(15) For every finite sequence p of elements of the carrier of $\mathbb{C}_{\rm F}$ holds $|\sum p| \leq \sum |p|$.

2. Operations on Polynomials

Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure, let p be a polynomial of L, and let n be a natural number. The functor p^n yields a sequence of L and is defined as follows:

(Def. 2) $p^n = \text{power}_{\text{Polynom-Ring}\,L}(p, n)$.

Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure, let p be a polynomial of L, and let n be a natural number. Note that p^n is finite-Support.

Next we state several propositions:

(16) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a polynomial of L. Then $p^0 = \mathbf{1}.L$.

(17) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a polynomial of L. Then $p^1 = p$.

(18) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a polynomial of L. Then $p^2 = p * p$.

(19) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a polynomial of L. Then $p^3 = p * p * p$.

(20) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure, p be a polynomial of L, and n be a natural number. Then $p^{n+1} = p^n * p$.

(21) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and n be a natural number. Then $(\mathbf{0}.L)^{n+1} = \mathbf{0}.L$.

(22) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and n be a natural number. Then $(\mathbf{1}.L)^n = \mathbf{1}.L$.

(23) Let L be a field, p be a polynomial of L, x be an element of L, and n be a natural number. Then $\mathrm{eval}(p^n, x) = \mathrm{power}_L(\mathrm{eval}(p, x), n)$.

(24) Let L be an integral domain and p be a polynomial of L. If $\mathrm{len}\, p \neq 0$, then for every natural number n holds $\mathrm{len}(p^n) = (n \cdot \mathrm{len}\, p - n) + 1$.

Let L be a non empty groupoid, let p be a sequence of L, and let v be an element of L. The functor $v \cdot p$ yielding a sequence of L is defined as follows:

(Def. 3) For every natural number n holds $(v \cdot p)(n) = v \cdot p(n)$.

Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure, let p be a polynomial of L, and let v be an element of L. Observe that $v \cdot p$ is finite-Support.

We now state several propositions:

(25) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and p be a polynomial of L. Then $\mathrm{len}(0_L \cdot p) = 0$.

(26) Let L be an add-associative right zeroed right complementable left unital commutative associative distributive field-like non empty double loop structure, p be a polynomial of L, and v be an element of L. If $v \neq 0_L$, then $\mathrm{len}(v \cdot p) = \mathrm{len}\, p$.

(27) Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure and p be a sequence of L. Then $0_L \cdot p = \mathbf{0}.L$.

(28) For every left unital non empty multiplicative loop structure L and for every sequence p of L holds $\mathbf{1}_L \cdot p = p$.

(29) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure and v be an element of L. Then $v \cdot \mathbf{0}.L = \mathbf{0}.L$.

(30) Let L be an add-associative right zeroed right complementable right unital right distributive non empty double loop structure and v be an element of L. Then $v \cdot \mathbf{1}.L = \langle_0 v\rangle$.

(31) Let L be an add-associative right zeroed right complementable left unital distributive commutative associative field-like non empty double loop structure, p be a polynomial of L, and v, x be elements of L. Then $\mathrm{eval}(v \cdot p, x) = v \cdot \mathrm{eval}(p, x)$.

(32) Let L be an add-associative right zeroed right complementable right distributive unital non empty double loop structure and p be a polynomial of L. Then $\mathrm{eval}(p, 0_L) = p(0)$.

Let L be a non empty zero structure and let z_0, z_1 be elements of L. The functor $\langle_0 z_0, z_1\rangle$ yields a sequence of L and is defined as follows:

(Def. 4) $\langle_0 z_0, z_1\rangle = \mathbf{0}.L +\cdot (0, z_0) +\cdot (1, z_1)$.

Next we state several propositions:

(33) Let L be a non empty zero structure and z_0 be an element of L. Then $\langle_0 z_0\rangle(0) = z_0$ and for every natural number n such that $n \geq 1$ holds $\langle_0 z_0\rangle(n) = 0_L$.

(34) For every non empty zero structure L and for every element z_0 of L such that $z_0 \neq 0_L$ holds $\mathrm{len}\langle_0 z_0\rangle = 1$.

(35) For every non empty zero structure L holds $\langle_0 0_L\rangle = \mathbf{0}.L$.

(36) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital integral domain-like non empty double loop structure and x, y be elements of L. Then $\langle_0 x\rangle * \langle_0 y\rangle = \langle_0 x \cdot y\rangle$.

(37) Let L be an Abelian add-associative right zeroed right complementable right unital associative commutative distributive field-like non empty double loop structure, x be an element of L, and n be a natural number. Then $\langle_0 x\rangle^n = \langle_0 \mathrm{power}_L(x, n)\rangle$.

(38) Let L be an add-associative right zeroed right complementable unital non empty double loop structure and z_0, x be elements of L. Then $\mathrm{eval}(\langle_0 z_0\rangle, x) = z_0$.

(39) Let L be a non empty zero structure and z_0, z_1 be elements of L. Then $\langle_0 z_0, z_1\rangle(0) = z_0$ and $\langle_0 z_0, z_1\rangle(1) = z_1$ and for every natural number n such that $n \geq 2$ holds $\langle_0 z_0, z_1\rangle(n) = 0_L$.

Let L be a non empty zero structure and let z_0, z_1 be elements of L. One can check that $\langle_0 z_0, z_1\rangle$ is finite-Support.

Next we state a number of propositions:

(40) For every non empty zero structure L and for all elements z_0, z_1 of L holds $\mathrm{len}\langle_0 z_0, z_1\rangle \leq 2$.

(41) For every non empty zero structure L and for all elements z_0, z_1 of L such that $z_1 \neq 0_L$ holds $\mathrm{len}\langle_0 z_0, z_1\rangle = 2$.

(42) For every non empty zero structure L and for every element z_0 of L such that $z_0 \neq 0_L$ holds $\mathrm{len}\langle_0 z_0, 0_L\rangle = 1$.

(43) For every non empty zero structure L holds $\langle_0 0_L, 0_L\rangle = \mathbf{0}.L$.

(44) For every non empty zero structure L and for every element z_0 of L holds $\langle_0 z_0, 0_L\rangle = \langle_0 z_0\rangle$.

(45) Let L be an add-associative right zeroed right complementable left distributive unital non empty double loop structure and z_0, z_1, x be elements of L. Then $\mathrm{eval}(\langle_0 z_0, z_1\rangle, x) = z_0 + z_1 \cdot x$.

(46) Let L be an add-associative right zeroed right complementable left distributive unital non empty double loop structure and z_0, z_1, x be elements of L. Then $\mathrm{eval}(\langle_0 z_0, 0_L\rangle, x) = z_0$.

(47) Let L be an add-associative right zeroed right complementable left distributive unital non empty double loop structure and z_0, z_1, x be elements of L. Then $\mathrm{eval}(\langle_0 0_L, z_1\rangle, x) = z_1 \cdot x$.

(48) Let L be an add-associative right zeroed right complementable left distributive well unital non empty double loop structure and z_0, z_1, x be elements of L. Then $\mathrm{eval}(\langle_0 z_0, \mathbf{1}_L\rangle, x) = z_0 + x$.

(49) Let L be an add-associative right zeroed right complementable left distributive well unital non empty double loop structure and z_0, z_1, x be elements of L. Then $\mathrm{eval}(\langle_0 0_L, \mathbf{1}_L\rangle, x) = x$.

3. Substitution in Polynomials

Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and let p, q be polynomials of L. The functor $p[q]$ yields a polynomial of L and is defined by the condition (Def. 5).

(Def. 5) There exists a finite sequence F of elements of the carrier of Polynom-Ring$L$ such that $p[q] = \sum F$ and $\mathrm{len} F = \mathrm{len}\, p$ and for every natural number n such that $n \in \mathrm{dom} F$ holds $F(n) = p(n -' 1) \cdot q^{n -' 1}$.

One can prove the following propositions:

(50) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a polynomial of L. Then $(\mathbf{0}.L)[p] = \mathbf{0}.L$.

(51) Let L be an Abelian add-associative right zeroed right complementable right unital commutative distributive non empty double loop structure and p be a polynomial of L. Then $p[\mathbf{0}.L] = \langle_0 p(0)\rangle$.

(52) Let L be an Abelian add-associative right zeroed right complementable right unital associative commutative distributive field-like non empty double loop structure, p be a polynomial of L, and x be an element of L. Then $\mathrm{len}(p[\langle_0 x\rangle]) \leq 1$.

(53) For every field L and for all polynomials p, q of L such that $\mathrm{len}\, p \neq 0$ and $\mathrm{len}\, q > 1$ holds $\mathrm{len}(p[q]) = (\mathrm{len}\, p \cdot \mathrm{len}\, q - \mathrm{len}\, p - \mathrm{len}\, q) + 2$.

(54) For every field L and for all polynomials p, q of L and for every element x of L holds $\mathrm{eval}(p[q], x) = \mathrm{eval}(p, \mathrm{eval}(q, x))$.

4. Fundamental Theorem of Algebra

Let L be a unital non empty double loop structure, let p be a polynomial of L, and let x be an element of L. We say that x is a root of p if and only if:

(Def. 6) $\mathrm{eval}(p, x) = 0_L$.

Let L be a unital non empty double loop structure and let p be a polynomial of L. We say that p has roots if and only if:

(Def. 7) There exists an element x of L such that x is a root of p.

The following proposition is true

(55) For every unital non empty double loop structure L holds $\mathbf{0}.L$ has roots.

Let L be a unital non empty double loop structure. Note that $\mathbf{0}.L$ has roots.
One can prove the following proposition

(56) For every unital non empty double loop structure L and for every element x of L holds x is a root of $\mathbf{0}.L$.

Let L be a unital non empty double loop structure. One can check that there exists a polynomial of L which has roots.
Let L be a unital non empty double loop structure. We say that L is algebraic-closed if and only if:

(Def. 8) For every polynomial p of L such that $\mathrm{len}\, p > 1$ holds p has roots.

Let L be a unital non empty double loop structure and let p be a polynomial of L. The functor $\mathrm{Roots}\, p$ yielding a subset of L is defined by:

(Def. 9) For every element x of L holds $x \in \mathrm{Roots}\, p$ iff x is a root of p.

Let L be a commutative associative left unital distributive field-like non empty double loop structure and let p be a polynomial of L. The functor $\mathrm{NormPolynomial}\, p$ yields a sequence of L and is defined by:

(Def. 10) For every natural number n holds $(\mathrm{NormPolynomial}\, p)(n) = \frac{p(n)}{p(\mathrm{len}\, p -' 1)}$.

Let L be an add-associative right zeroed right complementable commutative associative left unital distributive field-like non empty double loop structure and let p be a polynomial of L. One can check that $\mathrm{NormPolynomial}\, p$ is finite-Support.
The following propositions are true:

(57) Let L be a commutative associative left unital distributive field-like non empty double loop structure and p be a polynomial of L. If $\mathrm{len}\, p \neq 0$, then $(\mathrm{NormPolynomial}\, p)(\mathrm{len}\, p -' 1) = \mathbf{1}_L$.

(58) For every field L and for every polynomial p of L such that $\mathrm{len}\, p \neq 0$ holds $\mathrm{len}\, \mathrm{NormPolynomial}\, p = \mathrm{len}\, p$.

(59) Let L be a field and p be a polynomial of L. If $\operatorname{len} p \neq 0$, then for every element x of L holds $\operatorname{eval}(\operatorname{NormPolynomial} p, x) = \frac{\operatorname{eval}(p,x)}{p(\operatorname{len} p -' 1)}$.

(60) Let L be a field and p be a polynomial of L. Suppose $\operatorname{len} p \neq 0$. Let x be an element of L. Then x is a root of p if and only if x is a root of $\operatorname{NormPolynomial} p$.

(61) For every field L and for every polynomial p of L such that $\operatorname{len} p \neq 0$ holds p has roots iff $\operatorname{NormPolynomial} p$ has roots.

(62) For every field L and for every polynomial p of L such that $\operatorname{len} p \neq 0$ holds $\operatorname{Roots} p = \operatorname{Roots} \operatorname{NormPolynomial} p$.

(63) $\mathrm{id}_{\mathbb{C}}$ is continuous on $\mathbb{C}$.

(64) For every element x of $\mathbb{C}$ holds $\mathbb{C} \longmapsto x$ is continuous on $\mathbb{C}$.

Let L be a unital non empty groupoid, let x be an element of L, and let n be a natural number. The functor $\operatorname{FPower}(x,n)$ yielding a map from L into L is defined as follows:

(Def. 11) For every element y of L holds $(\operatorname{FPower}(x,n))(y) = x \cdot \operatorname{power}_L(y, n)$.

Next we state several propositions:

(65) For every unital non empty groupoid L holds $\operatorname{FPower}(1_L, 1) = \mathrm{id}_{\text{the carrier of } L}$.

(66) $\operatorname{FPower}(\mathbf{1}_{\mathbb{C}_{\mathrm{F}}}, 2) = \mathrm{id}_{\mathbb{C}}\, \mathrm{id}_{\mathbb{C}}$.

(67) For every unital non empty groupoid L and for every element x of L holds $\operatorname{FPower}(x,0) = (\text{the carrier of } L) \longmapsto x$.

(68) For every element x of $\mathbb{C}_{\mathrm{F}}$ there exists an element x_1 of $\mathbb{C}$ such that $x = x_1$ and $\operatorname{FPower}(x,1) = x_1\, \mathrm{id}_{\mathbb{C}}$.

(69) For every element x of $\mathbb{C}_{\mathrm{F}}$ there exists an element x_1 of $\mathbb{C}$ such that $x = x_1$ and $\operatorname{FPower}(x,2) = x_1\, (\mathrm{id}_{\mathbb{C}}\, \mathrm{id}_{\mathbb{C}})$.

(70) Let x be an element of $\mathbb{C}_{\mathrm{F}}$ and n be a natural number. Then there exists a function f from $\mathbb{C}$ into $\mathbb{C}$ such that $f = \operatorname{FPower}(x,n)$ and $\operatorname{FPower}(x,n+1) = f\, \mathrm{id}_{\mathbb{C}}$.

(71) Let x be an element of $\mathbb{C}_{\mathrm{F}}$ and n be a natural number. Then there exists a function f from $\mathbb{C}$ into $\mathbb{C}$ such that $f = \operatorname{FPower}(x,n)$ and f is continuous on $\mathbb{C}$.

Let L be a unital non empty double loop structure and let p be a polynomial of L. The functor $\text{Polynomial-Function}(L,p)$ yields a map from L into L and is defined as follows:

(Def. 12) For every element x of L holds $(\text{Polynomial-Function}(L,p))(x) = \operatorname{eval}(p,x)$.

We now state four propositions:

(72) For every polynomial p of $\mathbb{C}_{\mathrm{F}}$ there exists a function f from $\mathbb{C}$ into $\mathbb{C}$ such that $f = \text{Polynomial-Function}(\mathbb{C}_{\mathrm{F}}, p)$ and f is continuous on $\mathbb{C}$.

(73) Let p be a polynomial of $\mathbb{C}_{\mathrm{F}}$. Suppose $\operatorname{len} p > 2$ and $|p(\operatorname{len} p -' 1)| = 1$. Let F be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{len} F = \operatorname{len} p$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = |p(n -' 1)|$. Let z be an element of $\mathbb{C}_{\mathrm{F}}$. If $|z| > \sum F$, then $|\operatorname{eval}(p,z)| > |p(0)| + 1$.

(74) Let p be a polynomial of $\mathbb{C}_{\mathrm{F}}$. Suppose $\operatorname{len} p > 2$. Then there exists an element z_0 of $\mathbb{C}_{\mathrm{F}}$ such that for every element z of $\mathbb{C}_{\mathrm{F}}$ holds $|\operatorname{eval}(p,z)| \geq |\operatorname{eval}(p,z_0)|$.

(75) For every polynomial p of $\mathbb{C}_{\mathrm{F}}$ such that $\operatorname{len} p > 1$ holds p has roots.

Let us note that $\mathbb{C}_{\mathrm{F}}$ is algebraic-closed.

Let us note that there exists a left unital right unital non empty double loop structure which is algebraic-closed, add-associative, right zeroed, right complementable, Abelian, commutative, associative, distributive, field-like, and non degenerated.

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