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The Evaluation of Polynomials

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The articles [18], [25], [19], [1], [20], [26], [5], [6], [4], [2], [7], [24], [22], [9], [8], [14], [15], [16], [21], [10], [12], [27], [3], [23], [17], [13], and [11] provide the notation and terminology for this paper.

1. Preliminaries

The following propositions are true:

(1) For every natural number n holds $0 -' n = 0$.

(3)[1] Let D be a non empty set, p be a finite sequence of elements of D, and n be a natural number. If $1 \leq n$ and $n \leq \operatorname{len} p$, then $p = (p \restriction (n -' 1)) \frown \langle p(n) \rangle \frown (p_{\downharpoonleft n})$.

Let us observe that every left zeroed add-right-cancelable right distributive left unital commutative associative non empty double loop structure which is field-like is also integral domain-like.

Let us note that there exists a non empty double loop structure which is strict, Abelian, add-associative, right zeroed, right complementable, associative, commutative, distributive, well unital, integral domain-like, field-like, non degenerated, and non trivial.

2. About Polynomials

The following propositions are true:

(5)[2] Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure and p be a sequence of L. Then $\mathbf{0}.L * p = \mathbf{0}.L$.

(6) For every non empty zero structure L holds $\operatorname{len} \mathbf{0}.L = 0$.

(7) For every non degenerated non empty multiplicative loop with zero structure L holds $\operatorname{len} \mathbf{1}.L = 1$.

(8) For every non empty zero structure L and for every polynomial p of L such that $\operatorname{len} p = 0$ holds $p = \mathbf{0}.L$.

(9) Let L be a right zeroed non empty loop structure, p, q be polynomials of L, and n be a natural number. If $n \geq \operatorname{len} p$ and $n \geq \operatorname{len} q$, then $n \geq \operatorname{len}(p + q)$.

[1] The proposition (2) has been removed.
[2] The proposition (4) has been removed.

(10) Let L be an add-associative right zeroed right complementable non empty loop structure and p, q be polynomials of L. If $\operatorname{len} p \neq \operatorname{len} q$, then $\operatorname{len}(p+q) = \max(\operatorname{len} p, \operatorname{len} q)$.

(11) Let L be an add-associative right zeroed right complementable non empty loop structure and p be a polynomial of L. Then $\operatorname{len}(-p) = \operatorname{len} p$.

(12) Let L be an add-associative right zeroed right complementable non empty loop structure, p, q be polynomials of L, and n be a natural number. If $n \geq \operatorname{len} p$ and $n \geq \operatorname{len} q$, then $n \geq \operatorname{len}(p-q)$.

(13) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital non empty double loop structure and p, q be polynomials of L. If $p(\operatorname{len} p -' 1) \cdot q(\operatorname{len} q -' 1) \neq 0_L$, then $\operatorname{len}(p * q) = (\operatorname{len} p + \operatorname{len} q) - 1$.

3. Leading Monomials

Let L be a non empty zero structure and let p be a polynomial of L. The functor Leading-Monomial p yields a sequence of L and is defined by:

(Def. 1) $(\text{Leading-Monomial}\, p)(\operatorname{len} p -' 1) = p(\operatorname{len} p -' 1)$ and for every natural number n such that $n \neq \operatorname{len} p -' 1$ holds $(\text{Leading-Monomial}\, p)(n) = 0_L$.

We now state the proposition

(14) For every non empty zero structure L and for every polynomial p of L holds $\text{Leading-Monomial}\, p = \mathbf{0}.L +\cdot (\operatorname{len} p -' 1, p(\operatorname{len} p -' 1))$.

Let L be a non empty zero structure and let p be a polynomial of L. One can check that Leading-Monomial p is finite-Support.

Next we state several propositions:

(15) For every non empty zero structure L and for every polynomial p of L such that $\operatorname{len} p = 0$ holds $\text{Leading-Monomial}\, p = \mathbf{0}.L$.

(16) For every non empty zero structure L holds $\text{Leading-Monomial}\, \mathbf{0}.L = \mathbf{0}.L$.

(17) For every non degenerated non empty multiplicative loop with zero structure L holds $\text{Leading-Monomial}\, \mathbf{1}.L = \mathbf{1}.L$.

(18) For every non empty zero structure L and for every polynomial p of L holds $\operatorname{len} \text{Leading-Monomial}\, p = \operatorname{len} p$.

(19) Let L be an add-associative right zeroed right complementable non empty loop structure and p be a polynomial of L. Suppose $\operatorname{len} p \neq 0$. Then there exists a polynomial q of L such that $\operatorname{len} q < \operatorname{len} p$ and $p = q + \text{Leading-Monomial}\, p$ and for every natural number n such that $n < \operatorname{len} p - 1$ holds $q(n) = p(n)$.

4. Evaluation of Polynomials

Let L be a unital non empty double loop structure, let p be a polynomial of L, and let x be an element of the carrier of L. The functor $\operatorname{eval}(p,x)$ yields an element of L and is defined by the condition (Def. 2).

(Def. 2) There exists a finite sequence F of elements of the carrier of L such that $\operatorname{eval}(p,x) = \sum F$ and $\operatorname{len} F = \operatorname{len} p$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = p(n -' 1) \cdot \operatorname{power}_L(x, n -' 1)$.

The following propositions are true:

(20) For every unital non empty double loop structure L and for every element x of the carrier of L holds $\operatorname{eval}(\mathbf{0}.L, x) = 0_L$.

(21) Let L be a well unital add-associative right zeroed right complementable associative non degenerated non empty double loop structure and x be an element of the carrier of L. Then $\mathrm{eval}(\mathbf{1}.L,x) = \mathbf{1}_L$.

(22) Let L be an Abelian add-associative right zeroed right complementable unital left distributive non empty double loop structure, p, q be polynomials of L, and x be an element of the carrier of L. Then $\mathrm{eval}(p+q,x) = \mathrm{eval}(p,x)+\mathrm{eval}(q,x)$.

(23) Let L be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, p be a polynomial of L, and x be an element of the carrier of L. Then $\mathrm{eval}(-p,x) = -\mathrm{eval}(p,x)$.

(24) Let L be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, p, q be polynomials of L, and x be an element of the carrier of L. Then $\mathrm{eval}(p-q,x) = \mathrm{eval}(p,x)-\mathrm{eval}(q,x)$.

(25) Let L be an add-associative right zeroed right complementable right zeroed distributive unital non empty double loop structure, p be a polynomial of L, and x be an element of the carrier of L. Then $\mathrm{eval}(\text{Leading-Monomial}\, p,x) = p(\mathrm{len}\, p -' 1)\cdot \mathrm{power}_L(x, \mathrm{len}\, p -' 1)$.

(26) Let L be an add-associative right zeroed right complementable Abelian left unital distributive commutative associative non trivial non empty double loop structure, p, q be polynomials of L, and x be an element of the carrier of L. Then $\mathrm{eval}(\text{Leading-Monomial}\, p*q,x) = \mathrm{eval}(\text{Leading-Monomial}\, p,x)\cdot \mathrm{eval}(q,x)$.

(27) Let L be an add-associative right zeroed right complementable Abelian left unital distributive commutative associative non trivial non empty double loop structure, p, q be polynomials of L, and x be an element of the carrier of L. Then $\mathrm{eval}(p*q,x) = \mathrm{eval}(p,x)\cdot \mathrm{eval}(q,x)$.

5. Evaluation Homomorphism

Let L be an add-associative right zeroed right complementable distributive unital non empty double loop structure and let x be an element of the carrier of L. The functor Polynom-Evaluation(L,x) yielding a map from Polynom-Ring$L$ into L is defined by:

(Def. 3) For every polynomial p of L holds $(\text{Polynom-Evaluation}(L,x))(p) = \mathrm{eval}(p,x)$.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated non empty double loop structure and let x be an element of the carrier of L. Observe that Polynom-Evaluation(L,x) is unity-preserving.

Let L be an Abelian add-associative right zeroed right complementable distributive unital non empty double loop structure and let x be an element of the carrier of L. Observe that Polynom-Evaluation(L,x) is additive.

Let L be an add-associative right zeroed right complementable Abelian left unital distributive commutative associative non trivial non empty double loop structure and let x be an element of the carrier of L. One can verify that Polynom-Evaluation(L,x) is multiplicative.

Let L be an add-associative right zeroed right complementable Abelian left unital distributive commutative associative non degenerated non empty double loop structure and let x be an element of the carrier of L. One can check that Polynom-Evaluation(L,x) is ring homomorphism.

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