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# The Algebra of Polynomials[1]

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**Summary.** In this paper we define the algebra of formal power series and the algebra of polynomials over an arbitrary field and prove some properties of these structures. We also formulate and prove theorems showing some general properties of sequences. These preliminaries will be used for defining and considering linear functionals on the algebra of polynomials.

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The articles [15], [5], [21], [16], [13], [1], [22], [3], [4], [2], [19], [6], [18], [17], [7], [10], [14], [11], [12], [9], [8], and [20] provide the notation and terminology for this paper.

## 1. PRELIMINARIES

Let $F$ be a 1-sorted structure. We consider algebra structures over $F$ as extensions of double loop structure and vector space structure over $F$ as systems

⟨ a carrier, an addition, a multiplication, a zero, a unity, a left multiplication ⟩,

where the carrier is a set, the addition and the multiplication are binary operations on the carrier, the zero and the unity are elements of the carrier, and the left multiplication is a function from [: the carrier of $F$, the carrier :] into the carrier.

Let $L$ be a non empty double loop structure. Observe that there exists an algebra structure over $L$ which is strict and non empty.

Let $L$ be a non empty double loop structure and let $A$ be a non empty algebra structure over $L$. We say that $A$ is mix-associative if and only if:

(Def. 1) For every element $a$ of $L$ and for all elements $x$, $y$ of $A$ holds $a \cdot (x \cdot y) = (a \cdot x) \cdot y$.

Let $L$ be a non empty double loop structure. One can verify that there exists a non empty algebra structure over $L$ which is well unital, distributive, vector space-like, and mix-associative.

Let $L$ be a non empty double loop structure. An algebra of $L$ is a well unital distributive vector space-like mix-associative non empty algebra structure over $L$.

Next we state two propositions:

(1) For all sets $X$, $Y$ and for every function $f$ from $[: X, Y :]$ into $X$ holds $\mathrm{dom}\, f = [: X, Y :]$.

(2) For all sets $X$, $Y$ and for every function $f$ from $[: X, Y :]$ into $Y$ holds $\mathrm{dom}\, f = [: X, Y :]$.

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## 2. THE ALGEBRA OF FORMAL POWER SERIES

Let $L$ be a non empty double loop structure. The functor Formal-Series $L$ yielding a strict non empty algebra structure over $L$ is defined by the conditions (Def. 2).

(Def. 2)(i) For every set $x$ holds $x \in$ the carrier of Formal-Series $L$ iff $x$ is a sequence of $L$,

(ii) for all elements $x$, $y$ of Formal-Series $L$ and for all sequences $p$, $q$ of $L$ such that $x = p$ and $y = q$ holds $x + y = p + q$,

(iii) for all elements $x$, $y$ of Formal-Series $L$ and for all sequences $p$, $q$ of $L$ such that $x = p$ and $y = q$ holds $x \cdot y = p * q$,

(iv) for every element $a$ of $L$ and for every element $x$ of Formal-Series $L$ and for every sequence $p$ of $L$ such that $x = p$ holds $a \cdot x = a \cdot p$,

(v) $0_{\text{Formal-Series}\,L} = \mathbf{0}.L$, and

(vi) $\mathbf{1}_{\text{Formal-Series}\,L} = \mathbf{1}.L$.

Let $L$ be an Abelian non empty double loop structure. One can verify that Formal-Series $L$ is Abelian.

Let $L$ be an add-associative non empty double loop structure. Note that Formal-Series $L$ is add-associative.

Let $L$ be a right zeroed non empty double loop structure. One can verify that Formal-Series $L$ is right zeroed.

Let $L$ be an add-associative right zeroed right complementable non empty double loop structure. One can check that Formal-Series $L$ is right complementable.

Let $L$ be an Abelian add-associative right zeroed commutative non empty double loop structure. One can check that Formal-Series $L$ is commutative.

Let $L$ be an Abelian add-associative right zeroed right complementable unital associative distributive non empty double loop structure. Observe that Formal-Series $L$ is associative.

Let $L$ be an add-associative right zeroed right complementable right unital right distributive non empty double loop structure. One can verify that Formal-Series $L$ is right unital.

Let us note that there exists a non empty double loop structure which is add-associative, associative, right zeroed, left zeroed, right unital, left unital, right complementable, and distributive.

Next we state three propositions:

(3) For every non empty set $D$ and for every non empty finite sequence $f$ of elements of $D$ holds $f_{\restriction 1} = f_{\upharpoonright 1}$.

(4) For every non empty set $D$ and for every non empty finite sequence $f$ of elements of $D$ holds $f = \langle f(1) \rangle \frown (f_{\upharpoonright 1})$.

(5) Let $L$ be an add-associative right zeroed left unital right complementable left distributive non empty double loop structure and $p$ be a sequence of $L$. Then $\mathbf{1}.L * p = p$.

Let $L$ be an add-associative right zeroed right complementable left unital left distributive non empty double loop structure. Observe that Formal-Series $L$ is left unital.

Let $L$ be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure. Note that Formal-Series $L$ is right distributive and Formal-Series $L$ is left distributive.

The following four propositions are true:

(6) Let $L$ be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure, $a$ be an element of $L$, and $p$, $q$ be sequences of $L$. Then $a \cdot (p + q) = a \cdot p + a \cdot q$.

(7) Let $L$ be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure, $a$, $b$ be elements of $L$, and $p$ be a sequence of $L$. Then $(a + b) \cdot p = a \cdot p + b \cdot p$.

(8) Let $L$ be an associative non empty double loop structure, $a$, $b$ be elements of $L$, and $p$ be a sequence of $L$. Then $(a \cdot b) \cdot p = a \cdot (b \cdot p)$.

(9) Let $L$ be an associative left unital non empty double loop structure and $p$ be a sequence of $L$. Then (the unity of $L) \cdot p = p$.

Let $L$ be an Abelian add-associative associative right zeroed right complementable left unital distributive non empty double loop structure. One can check that Formal-Series$L$ is vector space-like.

We now state the proposition

(10) Let $L$ be an Abelian left zeroed add-associative associative right zeroed right complementable distributive non empty double loop structure, $a$ be an element of $L$, and $p$, $q$ be sequences of $L$. Then $a \cdot (p * q) = (a \cdot p) * q$.

Let $L$ be an Abelian left zeroed add-associative associative right zeroed right complementable distributive non empty double loop structure. One can verify that Formal-Series$L$ is mix-associative.

Let $L$ be a left zeroed right zeroed add-associative left unital right unital right complementable distributive non empty double loop structure. Note that Formal-Series$L$ is well unital.

Let $L$ be a 1-sorted structure and let $A$ be an algebra structure over $L$. An algebra structure over $L$ is said to be a subalgebra of $A$ if it satisfies the conditions (Def. 3).

(Def. 3)(i) The carrier of it $\subseteq$ the carrier of $A$,

(ii) $\mathbf{1}_{\rm it} = \mathbf{1}_A$,

(iii) $0_{\rm it} = 0_A$,

(iv) the addition of it $=$ (the addition of $A) \restriction [:$the carrier of it, the carrier of it$:]$,

(v) the multiplication of it $=$ (the multiplication of $A) \restriction [:$the carrier of it, the carrier of it$:]$, and

(vi) the left multiplication of it $=$ (the left multiplication of $A) \restriction [:$the carrier of $L$, the carrier of it$:]$.

The following propositions are true:

(11) For every 1-sorted structure $L$ holds every algebra structure $A$ over $L$ is a subalgebra of $A$.

(12) Let $L$ be a 1-sorted structure and $A$, $B$, $C$ be algebra structures over $L$. Suppose $A$ is a subalgebra of $B$ and $B$ is a subalgebra of $C$. Then $A$ is a subalgebra of $C$.

(13) Let $L$ be a 1-sorted structure and $A$, $B$ be algebra structures over $L$. Suppose $A$ is a subalgebra of $B$ and $B$ is a subalgebra of $A$. Then the algebra structure of $A =$ the algebra structure of $B$.

(14) Let $L$ be a 1-sorted structure and $A$, $B$ be algebra structures over $L$. Suppose the algebra structure of $A =$ the algebra structure of $B$. Then $A$ is a subalgebra of $B$ and $B$ is a subalgebra of $A$.

Let $L$ be a non empty 1-sorted structure. Observe that there exists an algebra structure over $L$ which is non empty and strict.

Let $L$ be a 1-sorted structure and let $B$ be an algebra structure over $L$. One can check that there exists a subalgebra of $B$ which is strict.

Let $L$ be a non empty 1-sorted structure and let $B$ be a non empty algebra structure over $L$. One can check that there exists a subalgebra of $B$ which is strict and non empty.

Let $L$ be a non empty groupoid, let $B$ be a non empty algebra structure over $L$, and let $A$ be a subset of $B$. We say that $A$ is operations closed if and only if:

(Def. 4) $A$ is linearly closed and for all elements $x$, $y$ of $B$ such that $x \in A$ and $y \in A$ holds $x \cdot y \in A$ and $\mathbf{1}_B \in A$ and $0_B \in A$.

The following propositions are true:

(15) Let $L$ be a non empty groupoid, $B$ be a non empty algebra structure over $L$, $A$ be a non empty subalgebra of $B$, $x$, $y$ be elements of $B$, and $x'$, $y'$ be elements of $A$. If $x = x'$ and $y = y'$, then $x + y = x' + y'$.

(16) Let $L$ be a non empty groupoid, $B$ be a non empty algebra structure over $L$, $A$ be a non empty subalgebra of $B$, $x$, $y$ be elements of $B$, and $x'$, $y'$ be elements of $A$. If $x = x'$ and $y = y'$, then $x \cdot y = x' \cdot y'$.

(17) Let $L$ be a non empty groupoid, $B$ be a non empty algebra structure over $L$, $A$ be a non empty subalgebra of $B$, $a$ be an element of $L$, $x$ be an element of $B$, and $x'$ be an element of $A$. If $x = x'$, then $a \cdot x = a \cdot x'$.

(19)[1] Let $L$ be a non empty groupoid, $B$ be a non empty algebra structure over $L$, and $A$ be a non empty subalgebra of $B$. Then there exists a subset $C$ of $B$ such that the carrier of $A = C$ and $C$ is operations closed.

(20) Let $L$ be a non empty groupoid, $B$ be a non empty algebra structure over $L$, and $A$ be a subset of $B$. Suppose $A$ is operations closed. Then there exists a strict subalgebra $C$ of $B$ such that the carrier of $C = A$.

(21) Let $L$ be a non empty groupoid, $B$ be a non empty algebra structure over $L$, $A$ be a non empty subset of $B$, and $X$ be a family of subsets of $B$. Suppose that for every set $Y$ holds $Y \in X$ iff $Y \subseteq$ the carrier of $B$ and there exists a subalgebra $C$ of $B$ such that $Y =$ the carrier of $C$ and $A \subseteq Y$. Then $\bigcap X$ is operations closed.

Let $L$ be a non empty groupoid, let $B$ be a non empty algebra structure over $L$, and let $A$ be a non empty subset of $B$. The functor GenAlg$A$ yields a strict non empty subalgebra of $B$ and is defined by the conditions (Def. 5).

(Def. 5)(i) $A \subseteq$ the carrier of GenAlg$A$, and

(ii) for every subalgebra $C$ of $B$ such that $A \subseteq$ the carrier of $C$ holds the carrier of GenAlg$A \subseteq$ the carrier of $C$.

We now state the proposition

(22) Let $L$ be a non empty groupoid, $B$ be a non empty algebra structure over $L$, and $A$ be a non empty subset of $B$. If $A$ is operations closed, then the carrier of GenAlg$A = A$.

## 3. The Algebra of Polynomials

Let $L$ be an add-associative right zeroed right complementable distributive non empty double loop structure. The functor Polynom-Algebra$L$ yielding a strict non empty algebra structure over $L$ is defined as follows:

(Def. 6) There exists a non empty subset $A$ of Formal-Series$L$ such that $A =$ the carrier of Polynom-Ring$L$ and Polynom-Algebra$L =$ GenAlg$A$.

Let $L$ be an add-associative right zeroed right complementable distributive non empty double loop structure. Observe that Polynom-Ring$L$ is loop-like.

The following propositions are true:

(23) Let $L$ be an add-associative right zeroed right complementable distributive non empty double loop structure and $A$ be a non empty subset of Formal-Series$L$. If $A =$ the carrier of Polynom-Ring$L$, then $A$ is operations closed.

(24) Let $L$ be an add-associative right zeroed right complementable distributive non empty double loop structure. Then the double loop structure of Polynom-Algebra$L =$ Polynom-Ring$L$.

[1] The proposition (18) has been removed.

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