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# Classical and Non-classical Pasch Configurations in Ordered Affine Planes[1]

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**Summary.** Several configuration axioms, which are commonly called in the literature "Pasch Axioms" are introduced; three of them were investigated by Szmielew and concern invariantability of the betweenness relation under parallel projections, and two other were introduced by Tarski. It is demonstrated that they all are consequences of the trapezium axiom, adopted to characterize ordered affine spaces.

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The articles [1] and [2] provide the notation and terminology for this paper.

We adopt the following convention: $O_1$ denotes an ordered affine space and $a$, $a'$, $b$, $b'$, $c$, $c'$, $d$, $d_1$, $d_2$, $p$, $p'$, $x$, $y$ denote elements of $O_1$.

Let us consider $O_1$. We say that $O_1$ satisfies inner invariancy of betweenness relation under parallel projections if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let given $a$, $b$, $c$, $d$, $p$. Suppose not $\mathbf{L}(p,b,c)$ and $p$ is a midpoint of $b$, $a$ and $\mathbf{L}(p,c,d)$ and $b,c \parallel d,a$. Then $p$ is a midpoint of $c$, $d$.

We introduce $O_1$ satisfies inner invariancy of betweenness relation under parallel projections as a synonym of $O_1$ satisfies inner invariancy of betweenness relation under parallel projections.

Let us consider $O_1$. We say that $O_1$ satisfies outer invariancy of betweenness relation under parallel projections if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let given $a$, $b$, $c$, $d$, $p$. Suppose $b$ is a midpoint of $p$, $c$ and $\mathbf{L}(p,a,d)$ and $a,b \parallel c,d$ and not $\mathbf{L}(p,a,b)$. Then $a$ is a midpoint of $p$, $d$.

We introduce $O_1$ satisfies outer invariancy of betweenness relation under parallel projections as a synonym of $O_1$ satisfies outer invariancy of betweenness relation under parallel projections.

Let us consider $O_1$. We say that $O_1$ satisfies general invariancy of betweenness relation under parallel projections if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let given $a$, $b$, $c$, $a'$, $b'$, $c'$. Suppose not $\mathbf{L}(a,b,a')$ and $a,a' \parallel b,b'$ and $a,a' \parallel c,c'$ and $b$ is a midpoint of $a$, $c$ and $\mathbf{L}(a',b',c')$. Then $b'$ is a midpoint of $a'$, $c'$.

We introduce $O_1$ satisfies general invariancy of betweenness relation under parallel projections as a synonym of $O_1$ satisfies general invariancy of betweenness relation under parallel projections.

Let us consider $O_1$. We say that $O_1$ satisfies outer form of Pasch' Axiom if and only if the condition (Def. 4) is satisfied.

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(Def. 4) Let given $a$, $b$, $c$, $d$, $x$, $y$. Suppose $b$ is a midpoint of $a$, $d$ and $x$ is a midpoint of $b$, $c$ and not $\mathbf{L}(a,b,c)$. Then there exists $y$ such that $y$ is a midpoint of $a$, $c$ and $x$ is a midpoint of $y$, $d$.

We introduce $O_1$ satisfies outer form of Pasch' Axiom as a synonym of $O_1$ satisfies outer form of Pasch' Axiom.

Let us consider $O_1$. We say that $O_1$ satisfies inner form of Pasch' Axiom if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let given $a$, $b$, $c$, $d$, $x$, $y$. Suppose $b$ is a midpoint of $a$, $d$ and $x$ is a midpoint of $a$, $c$ and not $\mathbf{L}(a,b,c)$. Then there exists $y$ such that $y$ is a midpoint of $b$, $c$ and $y$ is a midpoint of $x$, $d$.

We introduce $O_1$ satisfies inner form of Pasch' Axiom as a synonym of $O_1$ satisfies inner form of Pasch' Axiom.

Let us consider $O_1$. We say that $O_1$ is Fanoian if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let given $a$, $b$, $c$, $d$. Suppose $a,b \upuparrows c,d$ and $a,c \upuparrows b,d$ and not $\mathbf{L}(a,b,c)$. Then there exists $x$ such that $x$ is a midpoint of $a$, $d$ and $x$ is a midpoint of $b$, $c$.

We introduce $O_1$ satisfies Fano Axiom as a synonym of $O_1$ is Fanoian.

We now state a number of propositions:

(7)[1] If $b,p \upuparrows p,c$ and $p \neq c$ and $b \neq p$, then there exists $d$ such that $a,p \upuparrows p,d$ and $a,b \parallel c,d$ and $c \neq d$ and $p \neq d$.

(8) If $p,b \upuparrows p,c$ and $p \neq c$ and $b \neq p$, then there exists $d$ such that $p,a \upuparrows p,d$ and $a,b \parallel c,d$ and $c \neq d$.

(9) If $p,b \parallel p,c$ and $p \neq b$, then there exists $d$ such that $p,a \parallel p,d$ and $a,b \parallel c,d$.

(11)[2] If not $\mathbf{L}(p,a,b)$ and $\mathbf{L}(p,b,c)$ and $\mathbf{L}(p,a,d_1)$ and $\mathbf{L}(p,a,d_2)$ and $a,b \parallel c,d_1$ and $a,b \parallel c,d_2$, then $d_1 = d_2$.

(12) If not $\mathbf{L}(a,b,c)$ and $a,b \parallel c,d_1$ and $a,b \parallel c,d_2$ and $a,c \parallel b,d_1$ and $a,c \parallel b,d_2$, then $d_1 = d_2$.

(13) If not $\mathbf{L}(p,b,c)$ and $p$ is a midpoint of $b$, $a$ and $\mathbf{L}(p,c,d)$ and $b,c \parallel d,a$, then $p$ is a midpoint of $c$, $d$.

(14) $O_1$ satisfies inner invariancy of betweenness relation under parallel projections.

(15) If $b$ is a midpoint of $p$, $c$ and $\mathbf{L}(p,a,d)$ and $a,b \parallel c,d$ and not $\mathbf{L}(p,a,b)$, then $a$ is a midpoint of $p$, $d$.

(16) $O_1$ satisfies outer invariancy of betweenness relation under parallel projections.

(17) If not $\mathbf{L}(a,b,a')$ and $a,a' \parallel b,b'$ and $a,a' \parallel c,c'$ and $b$ is a midpoint of $a$, $c$ and $\mathbf{L}(a',b',c')$, then $b'$ is a midpoint of $a'$, $c'$.

(18) $O_1$ satisfies general invariancy of betweenness relation under parallel projections.

(19) If not $\mathbf{L}(p,a,b)$ and $a,p \upuparrows p,a'$ and $b,p \upuparrows p,b'$ and $a,b \parallel a',b'$, then $a,b \upuparrows b',a'$.

(20) If not $\mathbf{L}(p,a,a')$ and $p,a \upuparrows p,b$ and $p,a' \upuparrows p,b'$ and $a,a' \parallel b,b'$, then $a,a' \upuparrows b,b'$.

(21) If not $\mathbf{L}(p,a,b)$ and $p,a \parallel b,c$ and $p,b \parallel a,c$, then $p,a \upuparrows b,c$ and $p,b \upuparrows a,c$.

(22) If $c$ is a midpoint of $p$, $b$ and $c,d \upuparrows b,a$ and $p,d \upuparrows p,a$ and not $\mathbf{L}(p,a,b)$ and $p \neq c$, then $d$ is a midpoint of $p$, $a$.

(23) If $d$ is a midpoint of $p$, $a$ and $c,d \upuparrows b,a$ and $p,c \upuparrows p,b$ and not $\mathbf{L}(p,a,b)$ and $p \neq c$, then $c$ is a midpoint of $p$, $b$.

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[1] The propositions (1)–(6) have been removed.

[2] The proposition (10) has been removed.

(24) If not $\mathbf{L}(p,a,b)$ and $p,b \upuparrows p,c$ and $b,a \upuparrows c,d$ and $\mathbf{L}(a,p,d)$ and $p \neq d$, then $p$ is not a midpoint of $a$, $d$.

(25) If $p,b \upuparrows p,c$ and $b \neq p$, then there exists $x$ such that $p,a \upuparrows p,x$ and $b,a \upuparrows c,x$.

(26) If $c$ is a midpoint of $p$, $b$, then there exists $x$ such that $x$ is a midpoint of $p$, $a$ and $b,a \upuparrows c,x$.

(27) If $p \neq b$ and $b$ is a midpoint of $p$, $c$, then there exists $x$ such that $a$ is a midpoint of $p$, $x$ and $b,a \upuparrows c,x$.

(28) If not $\mathbf{L}(p,a,b)$ and $c$ is a midpoint of $p$, $b$, then there exists $x$ such that $x$ is a midpoint of $p$, $a$ and $a,b \upuparrows x,c$.

(29) There exists $x$ such that $a,x \upuparrows b,c$ and $a,b \upuparrows x,c$.

(30) If $a,b \upuparrows c,d$ and not $\mathbf{L}(a,b,c)$, then there exists $x$ such that $x$ is a midpoint of $a$, $d$ and $x$ is a midpoint of $b$, $c$.

(32)[3] $O_1$ satisfies Fano Axiom.

(33) If $a,b \parallel c,d$ and $a,c \parallel b,d$ and not $\mathbf{L}(a,b,c)$, then there exists $x$ such that $\mathbf{L}(x,a,d)$ and $\mathbf{L}(x,b,c)$.

(34) If $a,b \parallel c,d$ and $a,c \parallel b,d$ and not $\mathbf{L}(a,b,c)$ and $\mathbf{L}(p,a,d)$ and $\mathbf{L}(p,b,c)$, then not $\mathbf{L}(p,a,b)$.

(35) Suppose $b$ is a midpoint of $a$, $d$ and $x$ is a midpoint of $b$, $c$ and not $\mathbf{L}(a,b,c)$. Then there exists $y$ such that $y$ is a midpoint of $a$, $c$ and $x$ is a midpoint of $y$, $d$.

(36) $O_1$ satisfies outer form of Pasch' Axiom.

(37) Suppose $b$ is a midpoint of $a$, $d$ and $x$ is a midpoint of $a$, $c$ and not $\mathbf{L}(a,b,c)$. Then there exists $y$ such that $y$ is a midpoint of $b$, $c$ and $y$ is a midpoint of $x$, $d$.

(38) $O_1$ satisfies inner form of Pasch' Axiom.

(39) Suppose $a$ is a midpoint of $p$, $b$ and $p,a \upuparrows p',a'$ and not $\mathbf{L}(p,a,p')$ and $\mathbf{L}(p',a',b')$ and $p,p' \upuparrows a,a'$ and $p,p' \upuparrows b,b'$. Then $a'$ is a midpoint of $p'$, $b'$.

## REFERENCES


[1] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/analoaf.html`.

[2] Henryk Oryszczyszyn and Krzysztof Prażmowski. Ordered affine spaces defined in terms of directed parallelity — part I. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/diraf.html`.



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[3] The proposition (31) has been removed.