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Increasing and Continuous Ordinal Sequences

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Summary. Concatenation of two ordinal sequences, the mode of all ordinals belonging to a universe and the mode of sequences of them with length equal to the rank of the universe are introduced. Besides, the increasing and continuous transfinite sequences, the limes of ordinal sequences and the power of ordinals, and the fact that every increasing and continuous transfinite sequence has critical numbers (fixed points) are discussed.

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The articles [7], [5], [8], [9], [4], [1], [2], [3], and [6] provide the notation and terminology for this paper.

We use the following convention: p_1, f_1, p_2 denote sequences of ordinal numbers and A, B, C denote ordinal numbers.

Let L be a sequence of ordinal numbers. Note that last L is ordinal.

Next we state the proposition

(1) If $\mathrm{dom}\, f_1 = \mathrm{succ}\, A$, then last f_1 is the limit of f_1 and $\lim f_1 = \mathrm{last}\, f_1$.

Let f_1, p_2 be transfinite sequences. The functor $f_1 \frown p_2$ yielding a transfinite sequence is defined by:

(Def. 1) $\mathrm{dom}(f_1 \frown p_2) = \mathrm{dom}\, f_1 + \mathrm{dom}\, p_2$ and for every A such that $A \in \mathrm{dom}\, f_1$ holds $(f_1 \frown p_2)(A) = f_1(A)$ and for every A such that $A \in \mathrm{dom}\, p_2$ holds $(f_1 \frown p_2)(\mathrm{dom}\, f_1 + A) = p_2(A)$.

Let us consider f_1, p_2. Note that $f_1 \frown p_2$ is ordinal yielding.

We now state four propositions:

(3)[1] If A is the limit of p_2, then A is the limit of $f_1 \frown p_2$.

(4) If A is the limit of f_1, then $B + A$ is the limit of $B + f_1$.

(5) If A is the limit of f_1, then $A \cdot B$ is the limit of $f_1 \cdot B$.

(6) Suppose that
(i) $\mathrm{dom}\, f_1 = \mathrm{dom}\, p_2$,
(ii) B is the limit of f_1,
(iii) C is the limit of p_2, and
(iv) for every A such that $A \in \mathrm{dom}\, f_1$ holds $f_1(A) \subseteq p_2(A)$ or for every A such that $A \in \mathrm{dom}\, f_1$ holds $f_1(A) \in p_2(A)$.

Then $B \subseteq C$.

[1] The proposition (2) has been removed.

In the sequel f_2, f_3 are sequences of ordinal numbers.

The following propositions are true:

(7) Suppose $\operatorname{dom} f_2 = \operatorname{dom} f_1$ and $\operatorname{dom} f_1 = \operatorname{dom} f_3$ and A is the limit of f_2 and the limit of f_3 and for every A such that $A \in \operatorname{dom} f_1$ holds $f_2(A) \subseteq f_1(A)$ and $f_1(A) \subseteq f_3(A)$. Then A is the limit of f_1.

(8) If $\operatorname{dom} f_1 \neq \emptyset$ and $\operatorname{dom} f_1$ is a limit ordinal number and f_1 is increasing, then $\sup f_1$ is the limit of f_1 and $\lim f_1 = \sup f_1$.

(9) If f_1 is increasing and $A \subseteq B$ and $B \in \operatorname{dom} f_1$, then $f_1(A) \subseteq f_1(B)$.

(10) If f_1 is increasing and $A \in \operatorname{dom} f_1$, then $A \subseteq f_1(A)$.

(11) If p_1 is increasing, then ${p_1}^{-1}(A)$ is an ordinal number.

(12) If f_2 is increasing, then $f_3 \cdot f_2$ is a sequence of ordinal numbers.

(13) If f_2 is increasing and f_3 is increasing, then there exists p_1 such that $p_1 = f_2 \cdot f_3$ and p_1 is increasing.

(14) If f_2 is increasing and A is the limit of f_3 and $\sup \operatorname{rng} f_2 = \operatorname{dom} f_3$ and $f_1 = f_3 \cdot f_2$, then A is the limit of f_1.

(15) If p_1 is increasing, then $p_1 \restriction A$ is increasing.

(16) If p_1 is increasing and $\operatorname{dom} p_1$ is a limit ordinal number, then $\sup p_1$ is a limit ordinal number.

(17) If f_1 is increasing and continuous and p_2 is continuous and $p_1 = p_2 \cdot f_1$, then p_1 is continuous.

(18) If for every A such that $A \in \operatorname{dom} f_1$ holds $f_1(A) = C + A$, then f_1 is increasing.

(19) If $C \neq \emptyset$ and for every A such that $A \in \operatorname{dom} f_1$ holds $f_1(A) = A \cdot C$, then f_1 is increasing.

(20) If $A \neq \emptyset$, then $\emptyset^A = \emptyset$.

(21) Suppose $A \neq \emptyset$ and A is a limit ordinal number. Let given f_1. Suppose $\operatorname{dom} f_1 = A$ and for every B such that $B \in A$ holds $f_1(B) = C^B$. Then C^A is the limit of f_1.

(22) If $C \neq \emptyset$, then $C^A \neq \emptyset$.

(23) If $\mathbf{1} \in C$, then $C^A \in C^{\operatorname{succ} A}$.

(24) If $\mathbf{1} \in C$ and $A \in B$, then $C^A \in C^B$.

(25) If $\mathbf{1} \in C$ and for every A such that $A \in \operatorname{dom} f_1$ holds $f_1(A) = C^A$, then f_1 is increasing.

(26) Suppose $\mathbf{1} \in C$ and $A \neq \emptyset$ and A is a limit ordinal number. Let given f_1. If $\operatorname{dom} f_1 = A$ and for every B such that $B \in A$ holds $f_1(B) = C^B$, then $C^A = \sup f_1$.

(27) If $C \neq \emptyset$ and $A \subseteq B$, then $C^A \subseteq C^B$.

(28) If $A \subseteq B$, then $A^C \subseteq B^C$.

(29) If $\mathbf{1} \in C$ and $A \neq \emptyset$, then $\mathbf{1} \in C^A$.

(30) $C^{A+B} = C^B \cdot C^A$.

(31) $(C^A)^B = C^{B \cdot A}$.

(32) If $\mathbf{1} \in C$, then $A \subseteq C^A$.

The scheme *CriticalNumber* deals with a unary functor $\mathcal{F}$ yielding an ordinal number, and states that:

There exists A such that $\mathcal{F}(A) = A$

provided the parameter meets the following requirements:

- For all A, B such that $A \in B$ holds $\mathcal{F}(A) \in \mathcal{F}(B)$, and
- Let given A. Suppose $A \neq \emptyset$ and A is a limit ordinal number. Let given p_1. If $\operatorname{dom} f_1 = A$ and for every B such that $B \in A$ holds $p_1(B) = \mathcal{F}(B)$, then $\mathcal{F}(A)$ is the limit of p_1.

In the sequel W denotes a universal class.

Let us consider W. An ordinal number is called an ordinal of W if:

(Def. 2) It $\in W$.

A sequence of ordinal numbers is said to be a transfinite sequence of ordinals of W if:

(Def. 3) $\operatorname{dom}\mathrm{it} = \mathrm{On}\,W$ and $\operatorname{rng}\mathrm{it} \subseteq \mathrm{On}\,W$.

Let us consider W. Note that there exists an ordinal of W which is non empty.

In the sequel A_1, B_1 are ordinals of W and p_1 is a transfinite sequence of ordinals of W.

The scheme *UOS Lambda* deals with a universal class $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an ordinal of $\mathcal{A}$, and states that:

There exists a transfinite sequence p_1 of ordinals of $\mathcal{A}$ such that for every ordinal a of $\mathcal{A}$ holds $p_1(a) = \mathcal{F}(a)$

for all values of the parameters.

Let us consider W. The functor $\mathbf{0}_W$ yields an ordinal of W and is defined by:

(Def. 4) $\mathbf{0}_W = \emptyset$.

The functor $\mathbf{1}_W$ yielding a non empty ordinal of W is defined as follows:

(Def. 5) $\mathbf{1}_W = \mathbf{1}$.

Let us consider p_1, A_1. Then $p_1(A_1)$ is an ordinal of W.

Let us consider W and let p_3, p_4 be transfinite sequences of ordinals of W. Then $p_4 \cdot p_3$ is a transfinite sequence of ordinals of W.

The following proposition is true

(35)[2] $\mathbf{0}_W = \emptyset$ and $\mathbf{1}_W = \mathbf{1}$.

Let us consider W, A_1. Then $\operatorname{succ} A_1$ is a non empty ordinal of W. Let us consider B_1. Then $A_1 + B_1$ is an ordinal of W.

Let us consider W, A_1, B_1. Then $A_1 \cdot B_1$ is an ordinal of W.

One can prove the following three propositions:

(36) $A_1 \in \operatorname{dom} p_1$.

(37) If $\operatorname{dom} f_1 \in W$ and $\operatorname{rng} f_1 \subseteq W$, then $\sup f_1 \in W$.

(38) If p_1 is increasing and continuous and $\omega \in W$, then there exists A such that $A \in \operatorname{dom} p_1$ and $p_1(A) = A$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal2.html.

[3] Grzegorz Bancerek. Ordinal arithmetics. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/ordinal3.html.

[2] The propositions (33) and (34) have been removed.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[5] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[6] Bogdan Nowak and Grzegorz Bancerek. Universal classes. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/classes2.html.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[8] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[9] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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