JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Ordinal Arithmetics

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Summary. At the beginning the article contains some auxiliary theorems concerning the constructors defined in papers [1] and [2]. Next simple properties of addition and multiplication of ordinals are shown, e.g. associativity of addition. Addition and multiplication of a transfinite sequence of ordinals and a ordinal are also introduced here. The goal of the article is the proof that the distributivity of multiplication wrt addition and the associativity of multiplication hold. Additionally new binary functors of ordinals are introduced: subtraction, exact division, and remainder and some of their basic properties are presented.

MML Identifier: ORDINAL3.

WWW: http://mizar.org/JFM/Vol2/ordinal3.html

The articles [5], [6], [7], [3], [1], [4], and [2] provide the notation and terminology for this paper.

We follow the rules: f_1, p_1 are sequences of ordinal numbers, A, B, C, D are ordinal numbers, and X, Y are sets.

One can prove the following propositions:

(1) $X \subseteq \operatorname{succ} X$.

(2) If $\operatorname{succ} X \subseteq Y$, then $X \subseteq Y$.

(5)[1] $A \in B$ iff $\operatorname{succ} A \in \operatorname{succ} B$.

(6) If $X \subseteq A$, then $\bigcup X$ is an ordinal number.

(7) $\bigcup \operatorname{On} X$ is an ordinal number.

(8) If $X \subseteq A$, then $\operatorname{On} X = X$.

(9) $\operatorname{On}\{A\} = \{A\}$.

(10) If $A \neq \emptyset$, then $\emptyset \in A$.

(11) $\inf A = \emptyset$.

(12) $\inf\{A\} = A$.

(13) If $X \subseteq A$, then $\bigcap X$ is an ordinal number.

Let us consider A, B. Observe that $A \cup B$ is ordinal and $A \cap B$ is ordinal.

Next we state a number of propositions:

(15)[2] $A \cup B = A$ or $A \cup B = B$.

[1] The propositions (3) and (4) have been removed.
[2] The proposition (14) has been removed.

(16) $A \cap B = A$ or $A \cap B = B$.

(17) If $A \in \mathbf{1}$, then $A = \emptyset$.

(18) $\mathbf{1} = \{\emptyset\}$.

(19) If $A \subseteq \mathbf{1}$, then $A = \emptyset$ or $A = \mathbf{1}$.

(20) If $A \subseteq B$ or $A \in B$ and if $C \in D$, then $A + C \in B + D$.

(21) If $A \subseteq B$ and $C \subseteq D$, then $A + C \subseteq B + D$.

(22) If $A \in B$ and if $C \subseteq D$ and $D \neq \emptyset$ or $C \in D$, then $A \cdot C \in B \cdot D$.

(23) If $A \subseteq B$ and $C \subseteq D$, then $A \cdot C \subseteq B \cdot D$.

(24) If $B + C = B + D$, then $C = D$.

(25) If $B + C \in B + D$, then $C \in D$.

(26) If $B + C \subseteq B + D$, then $C \subseteq D$.

(27) $A \subseteq A + B$ and $B \subseteq A + B$.

(28) If $A \in B$, then $A \in B + C$ and $A \in C + B$.

(29) If $A + B = \emptyset$, then $A = \emptyset$ and $B = \emptyset$.

(30) If $A \subseteq B$, then there exists C such that $B = A + C$.

(31) If $A \in B$, then there exists C such that $B = A + C$ and $C \neq \emptyset$.

(32) If $A \neq \emptyset$ and A is a limit ordinal number, then $B + A$ is a limit ordinal number.

(33) $(A + B) + C = A + (B + C)$.

(34) If $A \cdot B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.

(35) If $A \in B$ and $C \neq \emptyset$, then $A \in B \cdot C$ and $A \in C \cdot B$.

(36) If $B \cdot A = C \cdot A$ and $A \neq \emptyset$, then $B = C$.

(37) If $B \cdot A \in C \cdot A$, then $B \in C$.

(38) If $B \cdot A \subseteq C \cdot A$ and $A \neq \emptyset$, then $B \subseteq C$.

(39) If $B \neq \emptyset$, then $A \subseteq A \cdot B$ and $A \subseteq B \cdot A$.

(41)[3] If $A \cdot B = \mathbf{1}$, then $A = \mathbf{1}$ and $B = \mathbf{1}$.

(42) If $A \in B + C$, then $A \in B$ or there exists D such that $D \in C$ and $A = B + D$.

Let us consider C, f_1. The functor $C + f_1$ yields a sequence of ordinal numbers and is defined as follows:

(Def. 2)[4] $\mathrm{dom}(C + f_1) = \mathrm{dom}\, f_1$ and for every A such that $A \in \mathrm{dom}\, f_1$ holds $(C + f_1)(A) = C + f_1(A)$.

The functor $f_1 + C$ yields a sequence of ordinal numbers and is defined as follows:

(Def. 3) $\mathrm{dom}(f_1 + C) = \mathrm{dom}\, f_1$ and for every A such that $A \in \mathrm{dom}\, f_1$ holds $(f_1 + C)(A) = f_1(A) + C$.

The functor $C \cdot f_1$ yields a sequence of ordinal numbers and is defined by:

(Def. 4) $\mathrm{dom}(C \cdot f_1) = \mathrm{dom}\, f_1$ and for every A such that $A \in \mathrm{dom}\, f_1$ holds $(C \cdot f_1)(A) = C \cdot f_1(A)$.

[3] The proposition (40) has been removed.
[4] The definition (Def. 1) has been removed.

The functor $f_1 \cdot C$ yielding a sequence of ordinal numbers is defined by:

(Def. 5) $\operatorname{dom}(f_1 \cdot C) = \operatorname{dom} f_1$ and for every A such that $A \in \operatorname{dom} f_1$ holds $(f_1 \cdot C)(A) = f_1(A) \cdot C$.

Next we state a number of propositions:

(47)[5] If $\emptyset \neq \operatorname{dom} f_1$ and $\operatorname{dom} f_1 = \operatorname{dom} p_1$ and for all A, B such that $A \in \operatorname{dom} f_1$ and $B = f_1(A)$ holds $p_1(A) = C + B$, then $\sup p_1 = C + \sup f_1$.

(48) If A is a limit ordinal number, then $A \cdot B$ is a limit ordinal number.

(49) If $A \in B \cdot C$ and B is a limit ordinal number, then there exists D such that $D \in B$ and $A \in D \cdot C$.

(50) Suppose $\emptyset \neq \operatorname{dom} f_1$ and $\operatorname{dom} f_1 = \operatorname{dom} p_1$ and $C \neq \emptyset$ and $\sup f_1$ is a limit ordinal number and for all A, B such that $A \in \operatorname{dom} f_1$ and $B = f_1(A)$ holds $p_1(A) = B \cdot C$. Then $\sup p_1 = \sup f_1 \cdot C$.

(51) If $\emptyset \neq \operatorname{dom} f_1$, then $\sup(C + f_1) = C + \sup f_1$.

(52) If $\emptyset \neq \operatorname{dom} f_1$ and $C \neq \emptyset$ and $\sup f_1$ is a limit ordinal number, then $\sup(f_1 \cdot C) = \sup f_1 \cdot C$.

(53) If $B \neq \emptyset$, then $\bigcup(A + B) = A + \bigcup B$.

(54) $(A + B) \cdot C = A \cdot C + B \cdot C$.

(55) If $A \neq \emptyset$, then there exist C, D such that $B = C \cdot A + D$ and $D \in A$.

(56) For all ordinal numbers C_1, D_1, C_2, D_2 such that $C_1 \cdot A + D_1 = C_2 \cdot A + D_2$ and $D_1 \in A$ and $D_2 \in A$ holds $C_1 = C_2$ and $D_1 = D_2$.

(57) Suppose $\mathbf{1} \in B$ and $A \neq \emptyset$ and A is a limit ordinal number. Let given f_1. If $\operatorname{dom} f_1 = A$ and for every C such that $C \in A$ holds $f_1(C) = C \cdot B$, then $A \cdot B = \sup f_1$.

(58) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

Let us consider A, B. The functor $A - B$ yields an ordinal number and is defined by:

(Def. 6)(i) $A = B + (A - B)$ if $B \subseteq A$,

(ii) $A - B = \emptyset$, otherwise.

The functor $A \div B$ yields an ordinal number and is defined as follows:

(Def. 7)(i) There exists C such that $A = (A \div B) \cdot B + C$ and $C \in B$ if $B \neq \emptyset$,

(ii) $A \div B = \emptyset$, otherwise.

Let us consider A, B. The functor $A \bmod B$ yields an ordinal number and is defined as follows:

(Def. 8) $A \bmod B = A - (A \div B) \cdot B$.

We now state a number of propositions:

(60)[6] If $A \in B$, then $B = A + (B - A)$.

(65)[7] $(A + B) - A = B$.

(66) If $A \in B$ and if $C \subseteq A$ or $C \in A$, then $A - C \in B - C$.

(67) $A - A = \emptyset$.

(68) If $A \in B$, then $B - A \neq \emptyset$ and $\emptyset \in B - A$.

[5] The propositions (43)–(46) have been removed.
[6] The proposition (59) has been removed.
[7] The propositions (61)–(64) have been removed.

(69) $A - \emptyset = A$ and $\emptyset - A = \emptyset$.

(70) $A - (B + C) = A - B - C$.

(71) If $A \subseteq B$, then $C - B \subseteq C - A$.

(72) If $A \subseteq B$, then $A - C \subseteq B - C$.

(73) If $C \neq \emptyset$ and $A \in B + C$, then $A - B \in C$.

(74) If $A + B \in C$, then $B \in C - A$.

(75) $A \subseteq B + (A - B)$.

(76) $A \cdot C - B \cdot C = (A - B) \cdot C$.

(77) $(A \div B) \cdot B \subseteq A$.

(78) $A = (A \div B) \cdot B + (A \bmod B)$.

(79) If $A = B \cdot C + D$ and $D \in C$, then $B = A \div C$ and $D = A \bmod C$.

(80) If $A \in B \cdot C$, then $A \div C \in B$ and $A \bmod C \in C$.

(81) If $B \neq \emptyset$, then $A \cdot B \div B = A$.

(82) $A \cdot B \bmod B = \emptyset$.

(83) $\emptyset \div A = \emptyset$ and $\emptyset \bmod A = \emptyset$ and $A \bmod \emptyset = A$.

(84) $A \div \mathbf{1} = A$ and $A \bmod \mathbf{1} = \emptyset$.

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Received March 1, 1990

Published January 2, 2004