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Sequences of Ordinal Numbers

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Summary. In the first part of the article we introduce the following operations: On X that yields the set of all ordinals which belong to the set X, Lim X that yields the set of all limit ordinals which belong to X, and inf X and sup X that yield the minimal ordinal belonging to X and the minimal ordinal greater than all ordinals belonging to X, respectively. The second part of the article starts with schemes that can be used to justify the correctness of definitions based on the transfinite induction (see [1] or [4]). The schemes are used to define addition, product and power of ordinal numbers. The operations of limes inferior and limes superior of sequences of ordinals are defined and the concepts of limit of ordinal sequence and increasing and continuous sequence are introduced.

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The articles [6], [3], [7], [8], [2], [1], and [5] provide the notation and terminology for this paper.

For simplicity, we follow the rules: A, A_1, A_2, B, C, D are ordinal numbers, X, Y are sets, x is a set, L is a transfinite sequence, and f is a function.

The scheme *Ordinal Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every A holds $\mathcal{P}[A]$

provided the following conditions are met:

- $\mathcal{P}[\emptyset]$,
- For every A such that $\mathcal{P}[A]$ holds $\mathcal{P}[\mathrm{succ}\,A]$, and
- For every A such that $A \neq \emptyset$ and A is a limit ordinal number and for every B such that $B \in A$ holds $\mathcal{P}[B]$ holds $\mathcal{P}[A]$.

One can prove the following propositions:

(1) $A \subseteq B$ iff $\mathrm{succ}\,A \subseteq \mathrm{succ}\,B$.

(2) $\bigcup \mathrm{succ}\,A = A$.

(3) $\mathrm{succ}\,A \subseteq 2^A$.

(4) $\emptyset$ is a limit ordinal number.

(5) $\bigcup A \subseteq A$.

Let us consider L. The functor $\mathrm{last}\,L$ yields a set and is defined as follows:

(Def. 1) $\mathrm{last}\,L = L(\bigcup \mathrm{dom}\,L)$.

We now state the proposition

(7)[1] If $\mathrm{dom}\,L = \mathrm{succ}\,A$, then $\mathrm{last}\,L = L(A)$.

[1] The proposition (6) has been removed.

Let us consider X. The functor $\mathrm{On}\,X$ yielding a set is defined by:

(Def. 2) $x \in \mathrm{On}\,X$ iff $x \in X$ and x is an ordinal number.

The functor $\mathrm{Lim}\,X$ yielding a set is defined as follows:

(Def. 3) $x \in \mathrm{Lim}\,X$ iff $x \in X$ and there exists A such that $x = A$ and A is a limit ordinal number.

Next we state several propositions:

(9)[2] $\mathrm{On}\,X \subseteq X$.

(10) $\mathrm{On}\,A = A$.

(11) If $X \subseteq Y$, then $\mathrm{On}\,X \subseteq \mathrm{On}\,Y$.

(13)[3] $\mathrm{Lim}\,X \subseteq X$.

(14) If $X \subseteq Y$, then $\mathrm{Lim}\,X \subseteq \mathrm{Lim}\,Y$.

(15) $\mathrm{Lim}\,X \subseteq \mathrm{On}\,X$.

(16) For every D there exists A such that $D \in A$ and A is a limit ordinal number.

(17) If for every x such that $x \in X$ holds x is an ordinal number, then $\bigcap X$ is an ordinal number.

The non empty ordinal number **1** is defined by:

(Def. 4) $\mathbf{1} = \mathrm{succ}\,\emptyset$.

The set ω is defined by the conditions (Def. 5).

(Def. 5)(i) $\emptyset \in \omega$,

(ii) ω is a limit ordinal number,

(iii) ω is ordinal, and

(iv) for every A such that $\emptyset \in A$ and A is a limit ordinal number holds $\omega \subseteq A$.

Let us note that ω is non empty and ordinal.
Let us consider X. The functor $\inf X$ yields an ordinal number and is defined by:

(Def. 6) $\inf X = \bigcap \mathrm{On}\,X$.

The functor $\sup X$ yields an ordinal number and is defined by:

(Def. 7) $\mathrm{On}\,X \subseteq \sup X$ and for every A such that $\mathrm{On}\,X \subseteq A$ holds $\sup X \subseteq A$.

Next we state a number of propositions:

(19)[4] $\emptyset \in \omega$ and ω is a limit ordinal number and for every A such that $\emptyset \in A$ and A is a limit ordinal number holds $\omega \subseteq A$.

(22)[5] If $A \in X$, then $\inf X \subseteq A$.

(23) If $\mathrm{On}\,X \neq \emptyset$ and for every A such that $A \in X$ holds $D \subseteq A$, then $D \subseteq \inf X$.

(24) If $A \in X$ and $X \subseteq Y$, then $\inf Y \subseteq \inf X$.

(25) If $A \in X$, then $\inf X \in X$.

(26) $\sup A = A$.

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[2] The proposition (8) has been removed.
[3] The proposition (12) has been removed.
[4] The proposition (18) has been removed.
[5] The propositions (20) and (21) have been removed.

(27) If $A \in X$, then $A \in \sup X$.

(28) If for every A such that $A \in X$ holds $A \in D$, then $\sup X \subseteq D$.

(29) If $A \in \sup X$, then there exists B such that $B \in X$ and $A \subseteq B$.

(30) If $X \subseteq Y$, then $\sup X \subseteq \sup Y$.

(31) $\sup\{A\} = \mathrm{succ}\, A$.

(32) $\inf X \subseteq \sup X$.

The scheme *TS Lambda* deals with an ordinal number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists L such that $\mathrm{dom}\, L = \mathcal{A}$ and for every A such that $A \in \mathcal{A}$ holds $L(A) = \mathcal{F}(A)$

for all values of the parameters.

Let us consider f. We say that f is ordinal yielding if and only if:

(Def. 8) There exists A such that $\mathrm{rng}\, f \subseteq A$.

Let us observe that there exists a transfinite sequence which is ordinal yielding.

A sequence of ordinal numbers is an ordinal yielding transfinite sequence.

Let us consider A. One can check that every transfinite sequence of elements of A is ordinal yielding.

Let L be a sequence of ordinal numbers and let us consider A. Note that $L \restriction A$ is ordinal yielding.

In the sequel f_1 is a sequence of ordinal numbers.

We now state the proposition

(34)[6] If $A \in \mathrm{dom}\, f_1$, then $f_1(A)$ is an ordinal number.

Let f be a sequence of ordinal numbers and let a be an ordinal number. Observe that $f(a)$ is ordinal.

Now we present a number of schemes. The scheme *OS Lambda* deals with an ordinal number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an ordinal number, and states that:

There exists f_1 such that $\mathrm{dom}\, f_1 = \mathcal{A}$ and for every A such that $A \in \mathcal{A}$ holds $f_1(A) = \mathcal{F}(A)$

for all values of the parameters.

The scheme *TS Uniq1* deals with an ordinal number $\mathcal{A}$, a set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{G}$ yielding a set, a transfinite sequence $\mathcal{C}$, and a transfinite sequence $\mathcal{D}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters meet the following conditions:

- $\mathrm{dom}\, \mathcal{C} = \mathcal{A}$,
- If $\emptyset \in \mathcal{A}$, then $\mathcal{C}(\emptyset) = \mathcal{B}$,
- For every A such that $\mathrm{succ}\, A \in \mathcal{A}$ holds $\mathcal{C}(\mathrm{succ}\, A) = \mathcal{F}(A, \mathcal{C}(A))$,
- For every A such that $A \in \mathcal{A}$ and $A \neq \emptyset$ and A is a limit ordinal number holds $\mathcal{C}(A) = \mathcal{G}(A, \mathcal{C} \restriction A)$,
- $\mathrm{dom}\, \mathcal{D} = \mathcal{A}$,
- If $\emptyset \in \mathcal{A}$, then $\mathcal{D}(\emptyset) = \mathcal{B}$,
- For every A such that $\mathrm{succ}\, A \in \mathcal{A}$ holds $\mathcal{D}(\mathrm{succ}\, A) = \mathcal{F}(A, \mathcal{D}(A))$, and
- For every A such that $A \in \mathcal{A}$ and $A \neq \emptyset$ and A is a limit ordinal number holds $\mathcal{D}(A) = \mathcal{G}(A, \mathcal{D} \restriction A)$.

The scheme *TS Exist1* deals with an ordinal number $\mathcal{A}$, a set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, and a binary functor $\mathcal{G}$ yielding a set, and states that:

There exists L such that

(i) $\mathrm{dom}\, L = \mathcal{A}$,

[6] The proposition (33) has been removed.

(ii) if $\emptyset \in \mathcal{A}$, then $L(\emptyset) = \mathcal{B}$,
(iii) for every A such that $\operatorname{succ} A \in \mathcal{A}$ holds $L(\operatorname{succ} A) = \mathcal{F}(A, L(A))$, and
(iv) for every A such that $A \in \mathcal{A}$ and $A \neq \emptyset$ and A is a limit ordinal number holds $L(A) = \mathcal{G}(A, L \restriction A)$

for all values of the parameters.

The scheme *TS Result* deals with a transfinite sequence $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, an ordinal number $\mathcal{B}$, a set $\mathcal{C}$, a binary functor $\mathcal{G}$ yielding a set, and a binary functor $\mathcal{H}$ yielding a set, and states that:

For every A such that $A \in \operatorname{dom} \mathcal{A}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$

provided the parameters meet the following conditions:

- Let given A, x. Then $x = \mathcal{F}(A)$ if and only if there exists L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\emptyset) = \mathcal{C}$ and for every C such that $\operatorname{succ} C \in \operatorname{succ} A$ holds $L(\operatorname{succ} C) = \mathcal{G}(C, L(C))$ and for every C such that $C \in \operatorname{succ} A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \mathcal{H}(C, L \restriction C)$,
- $\operatorname{dom} \mathcal{A} = \mathcal{B}$,
- If $\emptyset \in \mathcal{B}$, then $\mathcal{A}(\emptyset) = \mathcal{C}$,
- For every A such that $\operatorname{succ} A \in \mathcal{B}$ holds $\mathcal{A}(\operatorname{succ} A) = \mathcal{G}(A, \mathcal{A}(A))$, and
- For every A such that $A \in \mathcal{B}$ and $A \neq \emptyset$ and A is a limit ordinal number holds $\mathcal{A}(A) = \mathcal{H}(A, \mathcal{A} \restriction A)$.

The scheme *TS Def* deals with an ordinal number $\mathcal{A}$, a set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, and a binary functor $\mathcal{G}$ yielding a set, and states that:

(i) There exist x, L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} \mathcal{A}$ and $L(\emptyset) = \mathcal{B}$ and for every C such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ holds $L(\operatorname{succ} C) = \mathcal{F}(C, L(C))$ and for every C such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \mathcal{G}(C, L \restriction C)$, and

(ii) for all sets x_1, x_2 such that there exists L such that $x_1 = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} \mathcal{A}$ and $L(\emptyset) = \mathcal{B}$ and for every C such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ holds $L(\operatorname{succ} C) = \mathcal{F}(C, L(C))$ and for every C such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \mathcal{G}(C, L \restriction C)$ and there exists L such that $x_2 = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} \mathcal{A}$ and $L(\emptyset) = \mathcal{B}$ and for every C such that $\operatorname{succ} C \in \operatorname{succ} \mathcal{A}$ holds $L(\operatorname{succ} C) = \mathcal{F}(C, L(C))$ and for every C such that $C \in \operatorname{succ} \mathcal{A}$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \mathcal{G}(C, L \restriction C)$ holds $x_1 = x_2$

for all values of the parameters.

The scheme *TS Result0* deals with a unary functor $\mathcal{F}$ yielding a set, a set $\mathcal{A}$, a binary functor $\mathcal{G}$ yielding a set, and a binary functor $\mathcal{H}$ yielding a set, and states that:

$\mathcal{F}(\emptyset) = \mathcal{A}$

provided the following requirement is met:

- Let given A, x. Then $x = \mathcal{F}(A)$ if and only if there exists L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\emptyset) = \mathcal{A}$ and for every C such that $\operatorname{succ} C \in \operatorname{succ} A$ holds $L(\operatorname{succ} C) = \mathcal{G}(C, L(C))$ and for every C such that $C \in \operatorname{succ} A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \mathcal{H}(C, L \restriction C)$.

The scheme *TS ResultS* deals with a set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{G}$ yielding a set, and a unary functor $\mathcal{H}$ yielding a set, and states that:

For every A holds $\mathcal{H}(\operatorname{succ} A) = \mathcal{F}(A, \mathcal{H}(A))$

provided the parameters satisfy the following condition:

- Let given A, x. Then $x = \mathcal{H}(A)$ if and only if there exists L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\emptyset) = \mathcal{A}$ and for every C such that $\operatorname{succ} C \in \operatorname{succ} A$ holds $L(\operatorname{succ} C) = \mathcal{F}(C, L(C))$ and for every C such that $C \in \operatorname{succ} A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \mathcal{G}(C, L \restriction C)$.

The scheme *TS ResultL* deals with a transfinite sequence $\mathcal{A}$, an ordinal number $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, a set $\mathcal{C}$, a binary functor $\mathcal{G}$ yielding a set, and a binary functor $\mathcal{H}$ yielding a set, and states that:

$\mathcal{F}(\mathcal{B}) = \mathcal{H}(\mathcal{B}, \mathcal{A})$

provided the following conditions are satisfied:

- Let given A, x. Then $x = \mathcal{F}(A)$ if and only if there exists L such that $x = \operatorname{last} L$ and $\operatorname{dom} L = \operatorname{succ} A$ and $L(\emptyset) = \mathcal{C}$ and for every C such that $\operatorname{succ} C \in \operatorname{succ} A$ holds

$L(\mathrm{succ}\,C) = \mathcal{G}(C, L(C))$ and for every C such that $C \in \mathrm{succ}\,A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L(C) = \mathcal{H}(C, L \restriction C)$,

- $\mathcal{B} \neq \emptyset$ and $\mathcal{B}$ is a limit ordinal number,
- $\mathrm{dom}\,\mathcal{A} = \mathcal{B}$, and
- For every A such that $A \in \mathcal{B}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$.

The scheme *OS Exist* deals with an ordinal number $\mathcal{A}$, an ordinal number $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding an ordinal number, and a binary functor $\mathcal{G}$ yielding an ordinal number, and states that:

There exists f_1 such that

(i) $\mathrm{dom}\, f_1 = \mathcal{A}$,

(ii) if $\emptyset \in \mathcal{A}$, then $f_1(\emptyset) = \mathcal{B}$,

(iii) for every A such that $\mathrm{succ}\,A \in \mathcal{A}$ holds $f_1(\mathrm{succ}\,A) = \mathcal{F}(A, f_1(A))$, and

(iv) for every A such that $A \in \mathcal{A}$ and $A \neq \emptyset$ and A is a limit ordinal number holds $f_1(A) = \mathcal{G}(A, f_1 \restriction A)$

for all values of the parameters.

The scheme *OS Result* deals with a sequence $\mathcal{A}$ of ordinal numbers, a unary functor $\mathcal{F}$ yielding an ordinal number, an ordinal number $\mathcal{B}$, an ordinal number $\mathcal{C}$, a binary functor $\mathcal{G}$ yielding an ordinal number, and a binary functor $\mathcal{H}$ yielding an ordinal number, and states that:

For every A such that $A \in \mathrm{dom}\,\mathcal{A}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$

provided the parameters satisfy the following conditions:

- Let given A, B. Then $B = \mathcal{F}(A)$ if and only if there exists f_1 such that $B = \mathrm{last}\, f_1$ and $\mathrm{dom}\, f_1 = \mathrm{succ}\,A$ and $f_1(\emptyset) = \mathcal{C}$ and for every C such that $\mathrm{succ}\,C \in \mathrm{succ}\,A$ holds $f_1(\mathrm{succ}\,C) = \mathcal{G}(C, f_1(C))$ and for every C such that $C \in \mathrm{succ}\,A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $f_1(C) = \mathcal{H}(C, f_1 \restriction C)$,
- $\mathrm{dom}\,\mathcal{A} = \mathcal{B}$,
- If $\emptyset \in \mathcal{B}$, then $\mathcal{A}(\emptyset) = \mathcal{C}$,
- For every A such that $\mathrm{succ}\,A \in \mathcal{B}$ holds $\mathcal{A}(\mathrm{succ}\,A) = \mathcal{G}(A, \mathcal{A}(A))$, and
- For every A such that $A \in \mathcal{B}$ and $A \neq \emptyset$ and A is a limit ordinal number holds $\mathcal{A}(A) = \mathcal{H}(A, \mathcal{A} \restriction A)$.

The scheme *OS Def* deals with an ordinal number $\mathcal{A}$, an ordinal number $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding an ordinal number, and a binary functor $\mathcal{G}$ yielding an ordinal number, and states that:

(i) There exist A, f_1 such that $A = \mathrm{last}\, f_1$ and $\mathrm{dom}\, f_1 = \mathrm{succ}\,\mathcal{A}$ and $f_1(\emptyset) = \mathcal{B}$ and for every C such that $\mathrm{succ}\,C \in \mathrm{succ}\,\mathcal{A}$ holds $f_1(\mathrm{succ}\,C) = \mathcal{F}(C, f_1(C))$ and for every C such that $C \in \mathrm{succ}\,\mathcal{A}$ and $C \neq \emptyset$ and C is a limit ordinal number holds $f_1(C) = \mathcal{G}(C, f_1 \restriction C)$, and

(ii) for all A_1, A_2 such that there exists f_1 such that $A_1 = \mathrm{last}\, f_1$ and $\mathrm{dom}\, f_1 = \mathrm{succ}\,\mathcal{A}$ and $f_1(\emptyset) = \mathcal{B}$ and for every C such that $\mathrm{succ}\,C \in \mathrm{succ}\,\mathcal{A}$ holds $f_1(\mathrm{succ}\,C) = \mathcal{F}(C, f_1(C))$ and for every C such that $C \in \mathrm{succ}\,\mathcal{A}$ and $C \neq \emptyset$ and C is a limit ordinal number holds $f_1(C) = \mathcal{G}(C, f_1 \restriction C)$ and there exists f_1 such that $A_2 = \mathrm{last}\, f_1$ and $\mathrm{dom}\, f_1 = \mathrm{succ}\,\mathcal{A}$ and $f_1(\emptyset) = \mathcal{B}$ and for every C such that $\mathrm{succ}\,C \in \mathrm{succ}\,\mathcal{A}$ holds $f_1(\mathrm{succ}\,C) = \mathcal{F}(C, f_1(C))$ and for every C such that $C \in \mathrm{succ}\,\mathcal{A}$ and $C \neq \emptyset$ and C is a limit ordinal number holds $f_1(C) = \mathcal{G}(C, f_1 \restriction C)$ holds $A_1 = A_2$

for all values of the parameters.

The scheme *OS Result0* deals with a unary functor $\mathcal{F}$ yielding an ordinal number, an ordinal number $\mathcal{A}$, a binary functor $\mathcal{G}$ yielding an ordinal number, and a binary functor $\mathcal{H}$ yielding an ordinal number, and states that:

$\mathcal{F}(\emptyset) = \mathcal{A}$

provided the following condition is satisfied:

- Let given A, B. Then $B = \mathcal{F}(A)$ if and only if there exists f_1 such that $B = \mathrm{last}\, f_1$ and $\mathrm{dom}\, f_1 = \mathrm{succ}\,A$ and $f_1(\emptyset) = \mathcal{A}$ and for every C such that $\mathrm{succ}\,C \in \mathrm{succ}\,A$ holds $f_1(\mathrm{succ}\,C) = \mathcal{G}(C, f_1(C))$ and for every C such that $C \in \mathrm{succ}\,A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $f_1(C) = \mathcal{H}(C, f_1 \restriction C)$.

The scheme *OS ResultS* deals with an ordinal number $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding an ordinal number, a binary functor $\mathcal{G}$ yielding an ordinal number, and a unary functor $\mathcal{H}$ yielding an ordinal number, and states that:

For every A holds $\mathcal{H}(\mathrm{succ}\,A) = \mathcal{F}(A, \mathcal{H}(A))$

provided the following condition is met:

- Let given A, B. Then $B = \mathcal{H}(A)$ if and only if there exists f_1 such that $B = \operatorname{last} f_1$ and $\operatorname{dom} f_1 = \operatorname{succ} A$ and $f_1(\emptyset) = \mathcal{A}$ and for every C such that $\operatorname{succ} C \in \operatorname{succ} A$ holds $f_1(\operatorname{succ} C) = \mathcal{F}(C, f_1(C))$ and for every C such that $C \in \operatorname{succ} A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $f_1(C) = \mathcal{G}(C, f_1 \restriction C)$.

The scheme *OS ResultL* deals with a sequence $\mathcal{A}$ of ordinal numbers, an ordinal number $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an ordinal number, an ordinal number $\mathcal{C}$, a binary functor $\mathcal{G}$ yielding an ordinal number, and a binary functor $\mathcal{H}$ yielding an ordinal number, and states that:

$\mathcal{F}(\mathcal{B}) = \mathcal{H}(\mathcal{B}, \mathcal{A})$

provided the parameters meet the following conditions:

- Let given A, B. Then $B = \mathcal{F}(A)$ if and only if there exists f_1 such that $B = \operatorname{last} f_1$ and $\operatorname{dom} f_1 = \operatorname{succ} A$ and $f_1(\emptyset) = \mathcal{C}$ and for every C such that $\operatorname{succ} C \in \operatorname{succ} A$ holds $f_1(\operatorname{succ} C) = \mathcal{G}(C, f_1(C))$ and for every C such that $C \in \operatorname{succ} A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $f_1(C) = \mathcal{H}(C, f_1 \restriction C)$,
- $\mathcal{B} \neq \emptyset$ and $\mathcal{B}$ is a limit ordinal number,
- $\operatorname{dom} \mathcal{A} = \mathcal{B}$, and
- For every A such that $A \in \mathcal{B}$ holds $\mathcal{A}(A) = \mathcal{F}(A)$.

Let us consider L. The functor $\sup L$ yields an ordinal number and is defined by:

(Def. 9) $\sup L = \sup \operatorname{rng} L$.

The functor $\inf L$ yields an ordinal number and is defined by:

(Def. 10) $\inf L = \inf \operatorname{rng} L$.

The following proposition is true

(35) $\sup L = \sup \operatorname{rng} L$ and $\inf L = \inf \operatorname{rng} L$.

Let us consider L. The functor $\limsup L$ yielding an ordinal number is defined as follows:

(Def. 11) There exists f_1 such that $\limsup L = \inf f_1$ and $\operatorname{dom} f_1 = \operatorname{dom} L$ and for every A such that $A \in \operatorname{dom} L$ holds $f_1(A) = \sup \operatorname{rng}(L \restriction (\operatorname{dom} L \setminus A))$.

The functor $\liminf L$ yields an ordinal number and is defined as follows:

(Def. 12) There exists f_1 such that $\liminf L = \sup f_1$ and $\operatorname{dom} f_1 = \operatorname{dom} L$ and for every A such that $A \in \operatorname{dom} L$ holds $f_1(A) = \inf \operatorname{rng}(L \restriction (\operatorname{dom} L \setminus A))$.

Let us consider A, f_1. We say that A is the limit of f_1 if and only if:

(Def. 13)(i) There exists B such that $B \in \operatorname{dom} f_1$ and for every C such that $B \subseteq C$ and $C \in \operatorname{dom} f_1$ holds $f_1(C) = \emptyset$ if $A = \emptyset$,

(ii) for all B, C such that $B \in A$ and $A \in C$ there exists D such that $D \in \operatorname{dom} f_1$ and for every ordinal number E such that $D \subseteq E$ and $E \in \operatorname{dom} f_1$ holds $B \in f_1(E)$ and $f_1(E) \in C$, otherwise.

Let us consider f_1. Let us assume that there exists A such that A is the limit of f_1. The functor $\lim f_1$ yields an ordinal number and is defined by:

(Def. 14) $\lim f_1$ is the limit of f_1.

Let us consider A, f_1. The functor $\lim_{f_1} A$ yields an ordinal number and is defined as follows:

(Def. 15) $\lim_{f_1} A = \lim(f_1 \restriction A)$.

Let L be a sequence of ordinal numbers. We say that L is increasing if and only if:

(Def. 16) For all A, B such that $A \in B$ and $B \in \operatorname{dom} L$ holds $L(A) \in L(B)$.

We say that L is continuous if and only if:

(Def. 17) For all A, B such that $A \in \operatorname{dom} L$ and $A \neq \emptyset$ and A is a limit ordinal number and $B = L(A)$ holds B is the limit of $L \restriction A$.

Let us consider A, B. The functor $A+B$ yielding an ordinal number is defined by the condition (Def. 18).

(Def. 18) There exists f_1 such that

(i) $A+B = \operatorname{last} f_1$,

(ii) $\operatorname{dom} f_1 = \operatorname{succ} B$,

(iii) $f_1(\emptyset) = A$,

(iv) for every C such that $\operatorname{succ} C \in \operatorname{succ} B$ holds $f_1(\operatorname{succ} C) = \operatorname{succ} f_1(C)$, and

(v) for every C such that $C \in \operatorname{succ} B$ and $C \neq \emptyset$ and C is a limit ordinal number holds $f_1(C) = \sup(f_1 \restriction C)$.

Let us consider A, B. The functor $A \cdot B$ yielding an ordinal number is defined by the condition (Def. 19).

(Def. 19) There exists f_1 such that

(i) $A \cdot B = \operatorname{last} f_1$,

(ii) $\operatorname{dom} f_1 = \operatorname{succ} A$,

(iii) $f_1(\emptyset) = \emptyset$,

(iv) for every C such that $\operatorname{succ} C \in \operatorname{succ} A$ holds $f_1(\operatorname{succ} C) = f_1(C) + B$, and

(v) for every C such that $C \in \operatorname{succ} A$ and $C \neq \emptyset$ and C is a limit ordinal number holds $f_1(C) = \bigcup \sup(f_1 \restriction C)$.

Let us consider A, B. The functor A^B yielding an ordinal number is defined by the condition (Def. 20).

(Def. 20) There exists f_1 such that

(i) $A^B = \operatorname{last} f_1$,

(ii) $\operatorname{dom} f_1 = \operatorname{succ} B$,

(iii) $f_1(\emptyset) = \mathbf{1}$,

(iv) for every C such that $\operatorname{succ} C \in \operatorname{succ} B$ holds $f_1(\operatorname{succ} C) = A \cdot f_1(C)$, and

(v) for every C such that $C \in \operatorname{succ} B$ and $C \neq \emptyset$ and C is a limit ordinal number holds $f_1(C) = \lim(f_1 \restriction C)$.

We now state a number of propositions:

(44)[7] $A + \emptyset = A$.

(45) $A + \operatorname{succ} B = \operatorname{succ}(A+B)$.

(46) Suppose $B \neq \emptyset$ and B is a limit ordinal number. Let given f_1. If $\operatorname{dom} f_1 = B$ and for every C such that $C \in B$ holds $f_1(C) = A + C$, then $A + B = \sup f_1$.

(47) $\emptyset + A = A$.

(48) $A + \mathbf{1} = \operatorname{succ} A$.

(49) If $A \in B$, then $C + A \in C + B$.

(50) If $A \subseteq B$, then $C + A \subseteq C + B$.

(51) If $A \subseteq B$, then $A + C \subseteq B + C$.

(52) $\emptyset \cdot A = \emptyset$.

(53) $\operatorname{succ} B \cdot A = B \cdot A + A$.

[7] The propositions (36)–(43) have been removed.

(54) Suppose $B \neq \emptyset$ and B is a limit ordinal number. Let given f_1. If $\mathrm{dom}\, f_1 = B$ and for every C such that $C \in B$ holds $f_1(C) = C \cdot A$, then $B \cdot A = \bigcup \sup f_1$.

(55) $A \cdot \emptyset = \emptyset$.

(56) $\mathbf{1} \cdot A = A$ and $A \cdot \mathbf{1} = A$.

(57) If $C \neq \emptyset$ and $A \in B$, then $A \cdot C \in B \cdot C$.

(58) If $A \subseteq B$, then $A \cdot C \subseteq B \cdot C$.

(59) If $A \subseteq B$, then $C \cdot A \subseteq C \cdot B$.

(60) $A^{\emptyset} = \mathbf{1}$.

(61) $A^{\mathrm{succ}\, B} = A \cdot A^B$.

(62) Suppose $B \neq \emptyset$ and B is a limit ordinal number. Let given f_1. If $\mathrm{dom}\, f_1 = B$ and for every C such that $C \in B$ holds $f_1(C) = A^C$, then $A^B = \lim f_1$.

(63) $A^{\mathbf{1}} = A$ and $\mathbf{1}^A = \mathbf{1}$.

Let A be a set. We say that A is natural if and only if:

(Def. 21) $A \in \omega$.

The following proposition is true

(65)[8] For every A there exist B, C such that B is a limit ordinal number and C is natural and $A = B + C$.

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[8] The proposition (64) has been removed.