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Opposite Categories and Contravariant Functors

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Summary. The opposite category of a category, contravariant functors and duality functors are defined.

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The articles [6], [4], [7], [8], [1], [2], [5], and [3] provide the notation and terminology for this paper.

In this paper B, C, D are categories.

Let X, Y, Z be non empty sets and let f be a partial function from $[:X, Y:]$ to Z. Then $\curvearrowleft f$ is a partial function from $[:Y, X:]$ to Z.

Next we state the proposition

(1) $\langle$the objects of C, the morphisms of C, the cod-map of C, the dom-map of C, $\curvearrowleft$(the composition of C),the id-map of $C\rangle$ is a category.

Let us consider C. The functor $C^{\rm op}$ yielding a strict category is defined by the condition (Def. 1).

(Def. 1) $C^{\rm op} = \langle$the objects of C, the morphisms of C, the cod-map of C, the dom-map of C, $\curvearrowleft$(the composition of C),the id-map of $C\rangle$.

We now state the proposition

(2) $(C^{\rm op})^{\rm op} =$ the category structure of C.

Let us consider C and let c be an object of C. The functor $c^{\rm op}$ yielding an object of $C^{\rm op}$ is defined by:

(Def. 2) $c^{\rm op} = c$.

Let us consider C and let c be an object of $C^{\rm op}$. The functor ${}^{\rm op}c$ yields an object of C and is defined as follows:

(Def. 3) ${}^{\rm op}c = c^{\rm op}$.

The following propositions are true:

(3) For every object c of C holds $(c^{\rm op})^{\rm op} = c$.

(4) For every object c of C holds ${}^{\rm op}(c^{\rm op}) = c$.

(5) For every object c of $C^{\rm op}$ holds $({}^{\rm op}c)^{\rm op} = c$.

Let us consider C and let f be a morphism of C. The functor $f^{\rm op}$ yields a morphism of $C^{\rm op}$ and is defined as follows:

(Def. 4) $f^{\rm op} = f$.

Let us consider C and let f be a morphism of $C^{\rm op}$. The functor ${}^{\rm op}f$ yields a morphism of C and is defined by:

(Def. 5) ${}^{\rm op}f = f^{\rm op}$.

One can prove the following propositions:

(6) For every morphism f of C holds $(f^{\rm op})^{\rm op} = f$.

(7) For every morphism f of C holds ${}^{\rm op}(f^{\rm op}) = f$.

(8) For every morphism f of $C^{\rm op}$ holds $({}^{\rm op}f)^{\rm op} = f$.

(9) For every morphism f of C holds $\mathrm{dom}(f^{\rm op}) = \mathrm{cod}\, f$ and $\mathrm{cod}(f^{\rm op}) = \mathrm{dom}\, f$.

(10) For every morphism f of $C^{\rm op}$ holds $\mathrm{dom}\,{}^{\rm op}f = \mathrm{cod}\, f$ and $\mathrm{cod}\,{}^{\rm op}f = \mathrm{dom}\, f$.

(11) For every morphism f of C holds $(\mathrm{dom}\, f)^{\rm op} = \mathrm{cod}(f^{\rm op})$ and $(\mathrm{cod}\, f)^{\rm op} = \mathrm{dom}(f^{\rm op})$.

(12) For every morphism f of $C^{\rm op}$ holds ${}^{\rm op}\mathrm{dom}\, f = \mathrm{cod}\,{}^{\rm op}f$ and ${}^{\rm op}\mathrm{cod}\, f = \mathrm{dom}\,{}^{\rm op}f$.

(13) For all objects a, b of C and for every morphism f of C holds $f \in \mathrm{hom}(a,b)$ iff $f^{\rm op} \in \mathrm{hom}(b^{\rm op}, a^{\rm op})$.

(14) For all objects a, b of $C^{\rm op}$ and for every morphism f of $C^{\rm op}$ holds $f \in \mathrm{hom}(a,b)$ iff ${}^{\rm op}f \in \mathrm{hom}({}^{\rm op}b, {}^{\rm op}a)$.

(15) Let a, b be objects of C and f be a morphism from a to b. If $\mathrm{hom}(a,b) \neq \emptyset$, then $f^{\rm op}$ is a morphism from $b^{\rm op}$ to $a^{\rm op}$.

(16) Let a, b be objects of $C^{\rm op}$ and f be a morphism from a to b. If $\mathrm{hom}(a,b) \neq \emptyset$, then ${}^{\rm op}f$ is a morphism from ${}^{\rm op}b$ to ${}^{\rm op}a$.

(17) For all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $(g \cdot f)^{\rm op} = f^{\rm op} \cdot g^{\rm op}$.

(18) For all morphisms f, g of C such that $\mathrm{cod}(g^{\rm op}) = \mathrm{dom}(f^{\rm op})$ holds $(g \cdot f)^{\rm op} = f^{\rm op} \cdot g^{\rm op}$.

(19) For all morphisms f, g of $C^{\rm op}$ such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds ${}^{\rm op}(g \cdot f) = {}^{\rm op}f \cdot {}^{\rm op}g$.

(20) Let a, b, c be objects of C, f be a morphism from a to b, and g be a morphism from b to c. If $\mathrm{hom}(a,b) \neq \emptyset$ and $\mathrm{hom}(b,c) \neq \emptyset$, then $(g \cdot f)^{\rm op} = f^{\rm op} \cdot g^{\rm op}$.

(21) For every object a of C holds $(\mathrm{id}_a)^{\rm op} = \mathrm{id}_{a^{\rm op}}$.

(22) For every object a of $C^{\rm op}$ holds ${}^{\rm op}(\mathrm{id}_a) = \mathrm{id}_{{}^{\rm op}a}$.

(23) For every morphism f of C holds $f^{\rm op}$ is monic iff f is epi.

(24) For every morphism f of C holds $f^{\rm op}$ is epi iff f is monic.

(25) For every morphism f of C holds $f^{\rm op}$ is invertible iff f is invertible.

(26) For every object c of C holds c is initial iff $c^{\rm op}$ is terminal.

(27) For every object c of C holds $c^{\rm op}$ is initial iff c is terminal.

Let us consider C, B and let S be a function from the morphisms of $C^{\rm op}$ into the morphisms of B. The functor ${}_*S$ yielding a function from the morphisms of C into the morphisms of B is defined by:

(Def. 6) For every morphism f of C holds $({}_*S)(f) = S(f^{\rm op})$.

We now state three propositions:

(28) Let S be a function from the morphisms of $C^{\rm op}$ into the morphisms of B and f be a morphism of $C^{\rm op}$. Then $({}_*S)({}^{\rm op}f) = S(f)$.

(29) For every functor S from $C^{\rm op}$ to B and for every object c of C holds $({\rm Obj}\,{}_*S)(c) = ({\rm Obj}\,S)(c^{\rm op})$.

(30) For every functor S from $C^{\rm op}$ to B and for every object c of $C^{\rm op}$ holds $({\rm Obj}\,{}_*S)({}^{\rm op}c) = ({\rm Obj}\,S)(c)$.

Let us consider C, D. A function from the morphisms of C into the morphisms of D is said to be a contravariant functor from C into D if it satisfies the conditions (Def. 7).

(Def. 7)(i) For every object c of C there exists an object d of D such that ${\rm it}({\rm id}_c) = {\rm id}_d$,

(ii) for every morphism f of C holds ${\rm it}({\rm id}_{{\rm dom}\,f}) = {\rm id}_{{\rm cod}\,{\rm it}(f)}$ and ${\rm it}({\rm id}_{{\rm cod}\,f}) = {\rm id}_{{\rm dom}\,{\rm it}(f)}$, and

(iii) for all morphisms f, g of C such that ${\rm dom}\,g = {\rm cod}\,f$ holds ${\rm it}(g \cdot f) = {\rm it}(f) \cdot {\rm it}(g)$.

We now state several propositions:

(31) Let S be a contravariant functor from C into D, c be an object of C, and d be an object of D. If $S({\rm id}_c) = {\rm id}_d$, then $({\rm Obj}\,S)(c) = d$.

(32) For every contravariant functor S from C into D and for every object c of C holds $S({\rm id}_c) = {\rm id}_{({\rm Obj}\,S)(c)}$.

(33) For every contravariant functor S from C into D and for every morphism f of C holds $({\rm Obj}\,S)({\rm dom}\,f) = {\rm cod}\,S(f)$ and $({\rm Obj}\,S)({\rm cod}\,f) = {\rm dom}\,S(f)$.

(34) Let S be a contravariant functor from C into D and f, g be morphisms of C. If ${\rm dom}\,g = {\rm cod}\,f$, then ${\rm dom}\,S(f) = {\rm cod}\,S(g)$.

(35) For every functor S from $C^{\rm op}$ to B holds ${}_*S$ is a contravariant functor from C into B.

(36) Let S_1 be a contravariant functor from C into B and S_2 be a contravariant functor from B into D. Then $S_2 \cdot S_1$ is a functor from C to D.

(37) For every contravariant functor S from $C^{\rm op}$ into B and for every object c of C holds $({\rm Obj}\,{}_*S)(c) = ({\rm Obj}\,S)(c^{\rm op})$.

(38) For every contravariant functor S from $C^{\rm op}$ into B and for every object c of $C^{\rm op}$ holds $({\rm Obj}\,{}_*S)({}^{\rm op}c) = ({\rm Obj}\,S)(c)$.

(39) For every contravariant functor S from $C^{\rm op}$ into B holds ${}_*S$ is a functor from C to B.

Let us consider C, B and let S be a function from the morphisms of C into the morphisms of B. The functor ${}^*S$ yields a function from the morphisms of $C^{\rm op}$ into the morphisms of B and is defined by:

(Def. 8) For every morphism f of $C^{\rm op}$ holds $({}^*S)(f) = S({}^{\rm op}f)$.

The functor $\overline{S}$ yields a function from the morphisms of C into the morphisms of $B^{\rm op}$ and is defined by:

(Def. 9) For every morphism f of C holds $\overline{S}(f) = S(f)^{\rm op}$.

One can prove the following propositions:

(40) Let S be a function from the morphisms of C into the morphisms of B and f be a morphism of C. Then $({}^*S)(f^{\rm op}) = S(f)$.

(41) For every functor S from C to B and for every object c of $C^{\rm op}$ holds $({\rm Obj}\,{}^*S)(c) = ({\rm Obj}\,S)({}^{\rm op}c)$.

(42) For every functor S from C to B and for every object c of C holds $(\mathrm{Obj}\,{}^{*}S)(c^{\mathrm{op}}) = (\mathrm{Obj}\,S)(c)$.

(43) For every functor S from C to B and for every object c of C holds $(\mathrm{Obj}\,\overline{S})(c) = (\mathrm{Obj}\,S)(c)^{\mathrm{op}}$.

(44) For every contravariant functor S from C into B and for every object c of C^{op} holds $(\mathrm{Obj}\,{}^{*}S)(c) = (\mathrm{Obj}\,S)({}^{\mathrm{op}}c)$.

(45) For every contravariant functor S from C into B and for every object c of C holds $(\mathrm{Obj}\,{}^{*}S)(c^{\mathrm{op}}) = (\mathrm{Obj}\,S)(c)$.

(46) For every contravariant functor S from C into B and for every object c of C holds $(\mathrm{Obj}\,\overline{S})(c) = (\mathrm{Obj}\,S)(c)^{\mathrm{op}}$.

(47) Let F be a function from the morphisms of C into the morphisms of D and f be a morphism of C. Then $\overline{{}^{*}F}(f^{\mathrm{op}}) = F(f)^{\mathrm{op}}$.

(48) For every function S from the morphisms of C into the morphisms of D holds ${}_{*}({}^{*}S) = S$.

(49) For every function S from the morphisms of C^{op} into the morphisms of D holds ${}^{*}({}_{*}S) = S$.

(50) For every function S from the morphisms of C into the morphisms of D holds $\overline{{}^{*}S} = {}^{*}\overline{S}$.

(51) Let D be a strict category and S be a function from the morphisms of C into the morphisms of D. Then $\overline{\overline{S}} = S$.

(52) Let C be a strict category and S be a function from the morphisms of C into the morphisms of D. Then ${}^{*}({}^{*}S) = S$.

(53) Let S be a function from the morphisms of C into the morphisms of B and T be a function from the morphisms of B into the morphisms of D. Then ${}^{*}(T \cdot S) = T \cdot {}^{*}S$.

(54) Let S be a function from the morphisms of C into the morphisms of B and T be a function from the morphisms of B into the morphisms of D. Then $\overline{T \cdot S} = \overline{T} \cdot S$.

(55) Let F_1 be a function from the morphisms of C into the morphisms of B and F_2 be a function from the morphisms of B into the morphisms of D. Then $\overline{{}^{*}(F_2 \cdot F_1)} = \overline{{}^{*}F_2} \cdot \overline{{}^{*}F_1}$.

(56) For every contravariant functor S from C into D holds ${}^{*}S$ is a functor from C^{op} to D.

(57) For every contravariant functor S from C into D holds $\overline{S}$ is a functor from C to D^{op}.

(58) For every functor S from C to D holds ${}^{*}S$ is a contravariant functor from C^{op} into D.

(59) For every functor S from C to D holds $\overline{S}$ is a contravariant functor from C into D^{op}.

(60) Let S_1 be a contravariant functor from C into B and S_2 be a functor from B to D. Then $S_2 \cdot S_1$ is a contravariant functor from C into D.

(61) Let S_1 be a functor from C to B and S_2 be a contravariant functor from B into D. Then $S_2 \cdot S_1$ is a contravariant functor from C into D.

(62) For every functor F from C to D and for every object c of C holds $(\mathrm{Obj}\,\overline{{}^{*}F})(c^{\mathrm{op}}) = (\mathrm{Obj}\,F)(c)^{\mathrm{op}}$.

(63) For every contravariant functor F from C into D and for every object c of C holds $(\mathrm{Obj}\,\overline{{}^{*}F})(c^{\mathrm{op}}) = (\mathrm{Obj}\,F)(c)^{\mathrm{op}}$.

(64) For every functor F from C to D holds $\overline{{}^{*}F}$ is a functor from C^{op} to D^{op}.

(65) For every contravariant functor F from C into D holds $\overline{{}^{*}F}$ is a contravariant functor from C^{op} into D^{op}.

Let us consider C. The functor $\mathrm{id}^{\mathrm{op}}(C)$ yields a contravariant functor from C into C^{op} and is defined by:

(Def. 10) $\mathrm{id}^{\mathrm{op}}(C) = \overline{\mathrm{id}_C}$.

The functor ${}^{\mathrm{op}}\mathrm{id}(C)$ yielding a contravariant functor from C^{op} into C is defined by:

(Def. 11) ${}^{\mathrm{op}}\mathrm{id}(C) = {}^{*}(\mathrm{id}_C)$.

The following propositions are true:

(66) For every morphism f of C holds $\mathrm{id}^{\mathrm{op}}(C)(f) = f^{\mathrm{op}}$.

(67) For every object c of C holds $(\mathrm{Obj}\,\mathrm{id}^{\mathrm{op}}(C))(c) = c^{\mathrm{op}}$.

(68) For every morphism f of C^{op} holds $({}^{\mathrm{op}}\mathrm{id}(C))(f) = {}^{\mathrm{op}}f$.

(69) For every object c of C^{op} holds $(\mathrm{Obj}\,{}^{\mathrm{op}}\mathrm{id}(C))(c) = {}^{\mathrm{op}}c$.

(70) For every function S from the morphisms of C into the morphisms of D holds ${}^{*}S = S \cdot {}^{\mathrm{op}}\mathrm{id}(C)$ and $\overline{S} = \mathrm{id}^{\mathrm{op}}(D) \cdot S$.

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