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Representation Theorem for Heyting Lattices

Jolanta Kamieńska
Warsaw University
Białystok

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The articles [11], [7], [13], [1], [14], [5], [6], [4], [9], [10], [15], [16], [12], [2], [3], and [8] provide the notation and terminology for this paper.

Let us observe that every lower bound lattice which is Heyting is also implicative and every lattice which is implicative is also upper-bounded.

In the sequel T is a topological space and A, B are subsets of T.

We now state two propositions:

(1) $A \cap \mathrm{Int}(A^{\mathrm{c}} \cup B) \subseteq B$.

(2) For every subset C of T such that C is open and $A \cap C \subseteq B$ holds $C \subseteq \mathrm{Int}(A^{\mathrm{c}} \cup B)$.

Let T be a topological structure. The functor $\mathrm{Topology}(T)$ yields a family of subsets of T and is defined as follows:

(Def. 1) $\mathrm{Topology}(T) =$ the topology of T.

Let us consider T. Note that $\mathrm{Topology}(T)$ is non empty.

One can prove the following proposition

(3) For every subset A of T holds A is open iff $A \in \mathrm{Topology}(T)$.

Let T be a non empty topological space and let P, Q be elements of $\mathrm{Topology}(T)$. Then $P \cup Q$ is an element of $\mathrm{Topology}(T)$. Then $P \cap Q$ is an element of $\mathrm{Topology}(T)$.

In the sequel T denotes a non empty topological space and P, Q denote elements of $\mathrm{Topology}(T)$.

Let us consider T. The functor $\mathrm{TopUnion}(T)$ yielding a binary operation on $\mathrm{Topology}(T)$ is defined as follows:

(Def. 2) $(\mathrm{TopUnion}(T))(P, Q) = P \cup Q$.

The functor $\mathrm{TopMeet}(T)$ yielding a binary operation on $\mathrm{Topology}(T)$ is defined by:

(Def. 3) $(\mathrm{TopMeet}(T))(P, Q) = P \cap Q$.

One can prove the following proposition

(4) For every non empty topological space T holds $\langle \mathrm{Topology}(T), \mathrm{TopUnion}(T), \mathrm{TopMeet}(T) \rangle$ is a lattice.

Let us consider T. The functor $\mathrm{OpenSetLatt}(T)$ yielding a lattice is defined by:

(Def. 4) OpenSetLatt$(T) = \langle$Topology(T), TopUnion(T), TopMeet$(T)\rangle$.

The following proposition is true

(5) The carrier of OpenSetLatt$(T) =$ Topology(T).

In the sequel p, q denote elements of OpenSetLatt(T).
We now state several propositions:

(6) $p \sqcup q = p \cup q$ and $p \sqcap q = p \cap q$.

(7) $p \sqsubseteq q$ iff $p \subseteq q$.

(8) For all elements p', q' of Topology(T) such that $p = p'$ and $q = q'$ holds $p \sqsubseteq q$ iff $p' \subseteq q'$.

(9) OpenSetLatt(T) is implicative.

(10) OpenSetLatt(T) is lower-bounded and $\bot_{\mathrm{OpenSetLatt}(T)} = \emptyset$.

Let us consider T. Observe that OpenSetLatt(T) is Heyting.
The following proposition is true

(11) $\top_{\mathrm{OpenSetLatt}(T)} =$ the carrier of T.

For simplicity, we adopt the following convention: L is a distributive lattice, F is a filter of L, a, b are elements of L, and x, X_1, Y, Z are sets.
Let us consider L. The functor PrimeFilters(L) yields a set and is defined as follows:

(Def. 5) PrimeFilters$(L) = \{F : F \neq$ the carrier of $L \wedge F$is prime$\}$.

We now state the proposition

(12) $F \in$ PrimeFilters(L) iff $F \neq$ the carrier of L and F is prime.

Let us consider L. The functor StoneH(L) yields a function and is defined by:

(Def. 6) dom StoneH$(L) =$ the carrier of L and $(\mathrm{StoneH}(L))(a) = \{F : F \in \mathrm{PrimeFilters}(L) \wedge a \in F\}$.

Next we state two propositions:

(13) $F \in (\mathrm{StoneH}(L))(a)$ iff $F \in$ PrimeFilters(L) and $a \in F$.

(14) $x \in (\mathrm{StoneH}(L))(a)$ iff there exists F such that $F = x$ and $F \neq$ the carrier of L and F is prime and $a \in F$.

Let us consider L. The functor StoneS(L) yielding a set is defined as follows:

(Def. 7) StoneS$(L) =$ rng StoneH(L).

Let us consider L. Note that StoneS(L) is non empty.
One can prove the following three propositions:

(15) $x \in$ StoneS(L) iff there exists a such that $x = (\mathrm{StoneH}(L))(a)$.

(16) $(\mathrm{StoneH}(L))(a \sqcup b) = (\mathrm{StoneH}(L))(a) \cup (\mathrm{StoneH}(L))(b)$.

(17) $(\mathrm{StoneH}(L))(a \sqcap b) = (\mathrm{StoneH}(L))(a) \cap (\mathrm{StoneH}(L))(b)$.

Let us consider L, a. The functor Filters(a) yields a family of subsets of L and is defined as follows:

(Def. 8) Filters$(a) = \{F : a \in F\}$.

Let us consider L and let us consider a. Observe that Filters(a) is non empty.

The following propositions are true:

(18) $x \in$ Filters(a) iff x is a filter of L and $a \in x$.

(19) If $x \in$ Filters$(b) \setminus$ Filters(a), then x is a filter of L and $b \in x$ and $a \notin x$.

(20) Let given Z. Suppose $Z \neq \emptyset$ and $Z \subseteq$ Filters$(b) \setminus$ Filters(a) and Z is $\subseteq$-linear. Then there exists Y such that $Y \in$ Filters$(b) \setminus$ Filters(a) and for every X_1 such that $X_1 \in Z$ holds $X_1 \subseteq Y$.

(21) If $b \not\sqsubseteq a$, then $[b) \in$ Filters$(b) \setminus$ Filters(a).

(22) If $b \not\sqsubseteq a$, then there exists F such that $F \in$ PrimeFilters(L) and $a \notin F$ and $b \in F$.

(23) If $a \neq b$, then there exists F such that $F \in$ PrimeFilters(L).

(24) If $a \neq b$, then $($StoneH$(L))(a) \neq ($StoneH$(L))(b)$.

(25) StoneH(L) is one-to-one.

Let us consider L and let A, B be elements of StoneS(L). Then $A \cup B$ is an element of StoneS(L). Then $A \cap B$ is an element of StoneS(L).

Let us consider L. The functor SetUnion(L) yielding a binary operation on StoneS(L) is defined by:

(Def. 9) For all elements A, B of StoneS(L) holds $($SetUnion$(L))(A, B) = A \cup B$.

The functor SetMeet(L) yielding a binary operation on StoneS(L) is defined as follows:

(Def. 10) For all elements A, B of StoneS(L) holds $($SetMeet$(L))(A, B) = A \cap B$.

Next we state the proposition

(26) $\langle$StoneS(L), SetUnion(L), SetMeet$(L)\rangle$ is a lattice.

Let us consider L. The functor StoneLatt(L) yielding a lattice is defined by:

(Def. 11) StoneLatt$(L) = \langle$StoneS(L), SetUnion(L), SetMeet$(L)\rangle$.

In the sequel p, q are elements of StoneLatt(L).

One can prove the following propositions:

(27) For every L holds the carrier of StoneLatt$(L) =$ StoneS(L).

(28) $p \sqcup q = p \cup q$ and $p \sqcap q = p \cap q$.

(29) $p \sqsubseteq q$ iff $p \subseteq q$.

Let us consider L. Then StoneH(L) is a homomorphism from L to StoneLatt(L).

We now state three propositions:

(30) StoneH(L) is isomorphism.

(31) StoneLatt(L) is distributive.

(32) L and StoneLatt(L) are isomorphic.

Let us note that there exists a Heyting lattice which is non trivial.

In the sequel H is a non trivial Heyting lattice and p', q' are elements of H.

One can prove the following three propositions:

(33) $($StoneH$(H))(\top_H) =$ PrimeFilters(H).

(34) $($StoneH$(H))(\bot_H) = \emptyset$.

(35) $\mathrm{StoneS}(H) \subseteq 2^{\mathrm{PrimeFilters}(H)}$.

Let us consider H. Observe that $\mathrm{PrimeFilters}(H)$ is non empty.

Let us consider H. The functor $\mathrm{HTopSpace}(H)$ yielding a strict topological structure is defined by:

(Def. 12) The carrier of $\mathrm{HTopSpace}(H) = \mathrm{PrimeFilters}(H)$ and the topology of $\mathrm{HTopSpace}(H) = \{\bigcup A : A$ ranges over subsets of $\mathrm{StoneS}(H)\}$.

Let us consider H. Note that $\mathrm{HTopSpace}(H)$ is non empty and topological space-like.

We now state two propositions:

(36) The carrier of $\mathrm{OpenSetLatt}(\mathrm{HTopSpace}(H)) = \{\bigcup A : A$ ranges over subsets of $\mathrm{StoneS}(H)\}$.

(37) $\mathrm{StoneS}(H) \subseteq$ the carrier of $\mathrm{OpenSetLatt}(\mathrm{HTopSpace}(H))$.

Let us consider H. Then $\mathrm{StoneH}(H)$ is a homomorphism from H to $\mathrm{OpenSetLatt}(\mathrm{HTopSpace}(H))$.

We now state several propositions:

(38) $\mathrm{StoneH}(H)$ is monomorphism.

(39) $(\mathrm{StoneH}(H))(p' \Rightarrow q') = (\mathrm{StoneH}(H))(p') \Rightarrow (\mathrm{StoneH}(H))(q')$.

(40) $\mathrm{StoneH}(H)$ preserves implication.

(41) $\mathrm{StoneH}(H)$ preserves top.

(42) $\mathrm{StoneH}(H)$ preserves bottom.

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