JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Factorial and Newton Coefficients

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Summary. We define the following functions: exponential function (for natural exponent), factorial function and Newton coefficients. We prove some basic properties of notions introduced. There is also a proof of binominal formula. We prove also that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

MML Identifier: NEWTON.

WWW: http://mizar.org/JFM/Vol2/newton.html

The articles [9], [2], [3], [10], [8], [5], [4], [6], [1], and [7] provide the notation and terminology for this paper.

For simplicity, we follow the rules: i, k, n, m, l denote natural numbers, s, t, r denote natural numbers, a, b, x, y denote real numbers, and F, G denote finite sequences of elements of $\mathbb{R}$.

The following propositions are true:

(3)[1] For all finite sequences F, G such that $\operatorname{len} F = \operatorname{len} G$ and for every i such that $i \in \operatorname{dom} F$ holds $F(i) = G(i)$ holds $F = G$.

(5)[2] For every n such that $n \geq 1$ holds $\operatorname{Seg} n = \{1\} \cup \{k : 1 < k \wedge k < n\} \cup \{n\}$.

(6) For every F holds $\operatorname{len}(a \cdot F) = \operatorname{len} F$.

(7) $n \in \operatorname{dom} G$ iff $n \in \operatorname{dom}(a \cdot G)$.

Let i be a natural number and let x be a real number. Then $i \mapsto x$ is a finite sequence of elements of $\mathbb{R}$.

Let x be a real number and let n be a natural number. The functor x^n is defined by:

(Def. 1) $x^n = \prod(n \mapsto x)$.

Let x be a real number and let n be a natural number. Observe that x^n is real.

Let x be a real number and let n be a natural number. Then x^n is a real number.

We now state several propositions:

(9)[3] For every x holds $x^0 = 1$.

(10) For every x holds $x^1 = x$.

(11) For every s holds $x^{s+1} = x^s \cdot x$.

(12) $(x \cdot y)^s = x^s \cdot y^s$.

[1] The propositions (1) and (2) have been removed.
[2] The proposition (4) has been removed.
[3] The proposition (8) has been removed.

(13) $x^{s+t} = x^s \cdot x^t$.

(14) $(x^s)^t = x^{s \cdot t}$.

(15) For every s holds $1^s = 1$.

(16) For every s such that $s \geq 1$ holds $0^s = 0$.

Let n be a natural number. Then $\mathrm{idseq}(n)$ is a finite sequence of elements of $\mathbb{R}$.
Let n be a natural number. The functor $n!$ is defined as follows:

(Def. 2) $n! = \prod \mathrm{idseq}(n)$.

Let n be a natural number. Observe that $n!$ is real.
Let n be a natural number. Then $n!$ is a real number.
We now state several propositions:

(18)[4] $0! = 1$.

(19) $1! = 1$.

(20) $2! = 2$.

(21) For every s holds $(s+1)! = s! \cdot (s+1)$.

(22) For every s holds $s!$ is a natural number.

(23) For every s holds $s! > 0$.

(25)[5] For all s, t holds $s! \cdot t! \neq 0$.

Let k, n be natural numbers. The functor $\binom{n}{k}$ is defined by:

(Def. 3)(i) For every natural number l such that $l = n - k$ holds $\binom{n}{k} = \frac{n!}{k! \cdot l!}$ if $n \geq k$,

(ii) $\binom{n}{k} = 0$, otherwise.

Let k, n be natural numbers. Observe that $\binom{n}{k}$ is real.
Let k, n be natural numbers. Then $\binom{n}{k}$ is a real number.
The following propositions are true:

(27)[6] $\binom{0}{0} = 1$.

(29)[7] For every s holds $\binom{s}{0} = 1$.

(30) For all s, t such that $s \geq t$ and for every r such that $r = s - t$ holds $\binom{s}{t} = \binom{s}{r}$.

(31) For every s holds $\binom{s}{s} = 1$.

(32) For all s, t such that $s < t$ holds $\binom{t+1}{s+1} = \binom{t}{s+1} + \binom{t}{s}$ and $\binom{t+1}{s+1} = \binom{t}{s} + \binom{t}{s+1}$.

(33) For every s such that $s \geq 1$ holds $\binom{s}{1} = s$.

(34) For all s, t such that $s \geq 1$ and $t = s - 1$ holds $\binom{s}{t} = s$.

(35) For every s and for every r holds $\binom{s}{r}$ is a natural number.

(36) For all m, F such that $m \neq 0$ and $\mathrm{len}\, F = m$ and for all i, l such that $i \in \mathrm{dom}\, F$ and $l = (n+i) - 1$ holds $F(i) = \binom{l}{n}$ holds $\sum F = \binom{n+m}{n+1}$.

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[4] The proposition (17) has been removed.
[5] The proposition (24) has been removed.
[6] The proposition (26) has been removed.
[7] The proposition (28) has been removed.

Let a, b be real numbers and let n be a natural number. The functor $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{len}\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle = n+1$, and

(ii) for all natural numbers i, l, m such that $i \in \mathrm{dom}\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle$ and $m = i-1$ and $l = n-m$ holds $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle(i) = \binom{n}{m} \cdot a^l \cdot b^m$.

Next we state four propositions:

(38)[8] $\langle \binom{0}{0} a^0 b^0, \ldots, \binom{0}{0} a^0 b^0 \rangle = \langle 1 \rangle$.

(39) $\langle \binom{s}{0} a^0 b^s, \ldots, \binom{s}{s} a^s b^0 \rangle(1) = a^s$.

(40) $\langle \binom{s}{0} a^0 b^s, \ldots, \binom{s}{s} a^s b^0 \rangle(s+1) = b^s$.

(41) For every s holds $(a+b)^s = \sum \langle \binom{s}{0} a^0 b^s, \ldots, \binom{s}{s} a^s b^0 \rangle$.

Let n be a natural number. The functor $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 5) $\mathrm{len}\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle = n+1$ and for all natural numbers i, k such that $i \in \mathrm{dom}\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$ and $k = i-1$ holds $\langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle(i) = \binom{n}{k}$.

Next we state two propositions:

(43)[9] For every s holds $\langle \binom{s}{0}, \ldots, \binom{s}{s} \rangle = \langle \binom{s}{0} 1^0 1^s, \ldots, \binom{s}{s} 1^s 1^0 \rangle$.

(44) For every s holds $2^s = \sum \langle \binom{s}{0}, \ldots, \binom{s}{s} \rangle$.

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[8] The proposition (37) has been removed.
[9] The proposition (42) has been removed.

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Received July 27, 1990

Published January 2, 2004