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The Correspondence Between Homomorphisms of Universal Algebra & Many Sorted Algebra

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Summary. The aim of the article is to check the compatibility of the homomorphism of universal algebras introduced in [9] and the corresponding concept for many sorted algebras introduced in [10].

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The articles [14], [18], [19], [4], [20], [6], [5], [15], [1], [7], [11], [2], [3], [12], [8], [17], [9], [10], [13], and [16] provide the notation and terminology for this paper.

For simplicity, we follow the rules: U_1, U_2, U_3 are universal algebras, n is a natural number, A is a non empty set, and h is a function from U_1 into U_2.

Next we state four propositions:

(1) For all functions f, g and for every set C such that $\mathrm{rng}\, f \subseteq C$ holds $(g{\restriction}C) \cdot f = g \cdot f$.

(2) For every set I and for every subset C of I holds $C^* \subseteq I^*$.

(3) For every function f and for every set C such that f is function yielding holds $f{\restriction}C$ is function yielding.

(4) For every set I and for every subset C of I and for every many sorted set M indexed by I holds $(M{\restriction}C)^\# = M^\#{\restriction}C^*$.

Let us consider A, n and let a be an element of A. Then $n \mapsto a$ is a finite sequence of elements of A.

Let S, S' be non empty many sorted signatures. The predicate $S \leq S'$ is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of $S \subseteq$ the carrier of S',

(ii) the operation symbols of $S \subseteq$ the operation symbols of S',

(iii) (the arity of S')$\restriction$(the operation symbols of S) $=$ the arity of S, and

(iv) (the result sort of S')$\restriction$(the operation symbols of S) $=$ the result sort of S.

Let us note that the predicate $S \leq S'$ is reflexive.

One can prove the following propositions:

(5) For all non empty many sorted signatures S, S', S'' such that $S \leq S'$ and $S' \leq S''$ holds $S \leq S''$.

(6) For all strict non empty many sorted signatures S, S' such that $S \leq S'$ and $S' \leq S$ holds $S = S'$.

(7) Let g be a function, a be an element of A, and k be a natural number. If $1 \leq k$ and $k \leq n$, then $(a \dot{\longmapsto} g)((n \mapsto a)_k) = g$.

(8) Let I be a set, I_0 be a subset of I, A, B be many sorted sets indexed by I, F be a many sorted function from A into B, and A_0, B_0 be many sorted sets indexed by I_0. Suppose $A_0 = A{\restriction}I_0$ and $B_0 = B{\restriction}I_0$. Then $F{\restriction}I_0$ is a many sorted function from A_0 into B_0.

Let S, S' be strict non void non empty many sorted signatures and let A be a non-empty strict algebra over S'. Let us assume that $S \leq S'$. The functor $(A \, \mathrm{over} \, S)$ yields a non-empty strict algebra over S and is defined by the conditions (Def. 2).

(Def. 2)(i) The sorts of $(A \, \mathrm{over} \, S) =$ (the sorts of $A){\restriction}$(the carrier of S), and

(ii) the characteristics of $(A \, \mathrm{over} \, S) =$ (the characteristics of $A){\restriction}$(the operation symbols of S).

One can prove the following propositions:

(9) For every strict non void non empty many sorted signature S and for every non-empty strict algebra A over S holds $A = (A \, \mathrm{over} \, S)$.

(10) For all U_1, U_2 such that U_1 and U_2 are similar holds $\mathrm{MSSign}(U_1) = \mathrm{MSSign}(U_2)$.

Let U_1, U_2 be universal algebras and let h be a function from U_1 into U_2. Let us assume that $\mathrm{MSSign}(U_1) = \mathrm{MSSign}(U_2)$. The functor $\mathrm{MSAlg}(h)$ yields a many sorted function from $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2) \, \mathrm{over} \, \mathrm{MSSign}(U_1))$ and is defined as follows:

(Def. 3) $\mathrm{MSAlg}(h) = \{0\} \longmapsto h$.

We now state a number of propositions:

(11) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Let o be an operation symbol of $\mathrm{MSSign}(U_1)$. Then $(\mathrm{MSAlg}(h))$(the result sort of $o) = h$.

(12) For every operation symbol o of $\mathrm{MSSign}(U_1)$ holds $\mathrm{Den}(o, \mathrm{MSAlg}(U_1)) =$ (the characteristic of $U_1)(o)$.

(13) For every operation symbol o of $\mathrm{MSSign}(U_1)$ holds $\mathrm{Den}(o, \mathrm{MSAlg}(U_1))$ is an operation of U_1.

(14) For every operation symbol o of $\mathrm{MSSign}(U_1)$ holds every element of $\mathrm{Args}(o, \mathrm{MSAlg}(U_1))$ is a finite sequence of elements of the carrier of U_1.

(15) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Let o be an operation symbol of $\mathrm{MSSign}(U_1)$ and y be an element of $\mathrm{Args}(o, \mathrm{MSAlg}(U_1))$. Then $\mathrm{MSAlg}(h)\#y = h \cdot y$.

(16) If h is a homomorphism of U_1 into U_2, then $\mathrm{MSAlg}(h)$ is a homomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2) \, \mathrm{over} \, \mathrm{MSSign}(U_1))$.

(17) If U_1 and U_2 are similar, then $\mathrm{MSAlg}(h)$ is a many sorted set indexed by $\{0\}$.

(18) If h is an epimorphism of U_1 onto U_2, then $\mathrm{MSAlg}(h)$ is an epimorphism of $\mathrm{MSAlg}(U_1)$ onto $(\mathrm{MSAlg}(U_2) \, \mathrm{over} \, \mathrm{MSSign}(U_1))$.

(19) If h is a monomorphism of U_1 into U_2, then $\mathrm{MSAlg}(h)$ is a monomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2) \, \mathrm{over} \, \mathrm{MSSign}(U_1))$.

(20) If h is an isomorphism of U_1 and U_2, then $\mathrm{MSAlg}(h)$ is an isomorphism of $\mathrm{MSAlg}(U_1)$ and $(\mathrm{MSAlg}(U_2) \, \mathrm{over} \, \mathrm{MSSign}(U_1))$.

(21) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is a homomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is a homomorphism of U_1 into U_2.

(22) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is an epimorphism of $\mathrm{MSAlg}(U_1)$ onto $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is an epimorphism of U_1 onto U_2.

(23) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is a monomorphism of $\mathrm{MSAlg}(U_1)$ into $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is a monomorphism of U_1 into U_2.

(24) Let given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose $\mathrm{MSAlg}(h)$ is an isomorphism of $\mathrm{MSAlg}(U_1)$ and $(\mathrm{MSAlg}(U_2)\,\mathrm{over}\,\mathrm{MSSign}(U_1))$. Then h is an isomorphism of U_1 and U_2.

(25) $\mathrm{MSAlg}(\mathrm{id}_{\text{the carrier of } U_1}) = \mathrm{id}_{\text{the sorts of } \mathrm{MSAlg}(U_1)}$.

(26) Let given U_1, U_2, U_3. Suppose U_1 and U_2 are similar and U_2 and U_3 are similar. Let h_1 be a function from U_1 into U_2 and h_2 be a function from U_2 into U_3. Then $\mathrm{MSAlg}(h_2) \circ \mathrm{MSAlg}(h_1) = \mathrm{MSAlg}(h_2 \cdot h_1)$.

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