JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
Inst. of Computer Science, Univ. of Białystok

More on the Lattice of Many Sorted Equivalence Relations

Robert Milewski
Warsaw University
Białystok

MML Identifier: MSUALG_7.

WWW: http://mizar.org/JFM/Vol8/msualg_7.html

The articles [18], [6], [20], [21], [3], [5], [4], [17], [16], [19], [22], [1], [8], [7], [13], [2], [11], [12], [14], [15], [9], and [10] provide the notation and terminology for this paper.

1. Lattice of Many Sorted Equivalence Relations is Complete

For simplicity, we follow the rules: I denotes a non empty set, M denotes a many sorted set indexed by I, x denotes a set, and r_1, r_2 denote real numbers.

Next we state three propositions:

(1) For every set X holds $x \in$ the carrier of $\mathrm{EqRelLatt}(X)$ iff x is an equivalence relation of X.

(2) id_M is an equivalence relation of M.

(3) $[\![M,M]\!]$ is an equivalence relation of M.

Let us consider I, M. Observe that $\mathrm{EqRelLatt}(M)$ is bounded.

The following propositions are true:

(4) $\perp_{\mathrm{EqRelLatt}(M)} = \mathrm{id}_M$.

(5) $\top_{\mathrm{EqRelLatt}(M)} = [\![M,M]\!]$.

(6) Every subset of $\mathrm{EqRelLatt}(M)$ is a family of many sorted subsets indexed by $[\![M,M]\!]$.

(7) Let a, b be elements of $\mathrm{EqRelLatt}(M)$ and A, B be equivalence relations of M. If $a = A$ and $b = B$, then $a \sqsubseteq b$ iff $A \subseteq B$.

(8) Let X be a subset of $\mathrm{EqRelLatt}(M)$ and X_1 be a family of many sorted subsets indexed by $[\![M,M]\!]$. Suppose $X_1 = X$. Let a, b be equivalence relations of M. If $a = \bigcap|{:}X_1{:}|$ and $b \in X$, then $a \subseteq b$.

(9) Let X be a subset of $\mathrm{EqRelLatt}(M)$ and X_1 be a family of many sorted subsets indexed by $[\![M,M]\!]$. If $X_1 = X$ and X is non empty, then $\bigcap|{:}X_1{:}|$ is an equivalence relation of M.

Let L be a non empty lattice structure. Let us observe that L is complete if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let X be a subset of L. Then there exists an element a of L such that $X \sqsubseteq a$ and for every element b of L such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

Next we state the proposition

(10) $\mathrm{EqRelLatt}(M)$ is complete.

Let us consider I, M. One can check that $\mathrm{EqRelLatt}(M)$ is complete.
The following proposition is true

(11) Let X be a subset of $\mathrm{EqRelLatt}(M)$ and X_1 be a family of many sorted subsets indexed by $[\![M,M]\!]$. Suppose $X_1 = X$ and X is non empty. Let a, b be equivalence relations of M. If $a = \bigcap|{:}X_1{:}|$ and $b = \bigsqcap_{\mathrm{EqRelLatt}(M)}X$, then $a = b$.

2. SUBLATTICES INHERITING SUP'S AND INF'S

Let L be a lattice and let I_1 be a sublattice of L. We say that I_1 is $\bigsqcap$-inheriting if and only if:

(Def. 2) For every subset X of I_1 holds $\bigsqcap_L X \in$ the carrier of I_1.

We say that I_1 is $\bigsqcup$-inheriting if and only if:

(Def. 3) For every subset X of I_1 holds $\bigsqcup_L X \in$ the carrier of I_1.

Next we state several propositions:

(12) Let L be a lattice, L' be a sublattice of L, a, b be elements of L, and a', b' be elements of L'. If $a = a'$ and $b = b'$, then $a \sqcup b = a' \sqcup b'$ and $a \sqcap b = a' \sqcap b'$.

(13) Let L be a lattice, L' be a sublattice of L, X be a subset of L', a be an element of L, and a' be an element of L'. If $a = a'$, then $a \sqsubseteq X$ iff $a' \sqsubseteq X$.

(14) Let L be a lattice, L' be a sublattice of L, X be a subset of L', a be an element of L, and a' be an element of L'. If $a = a'$, then $X \sqsubseteq a$ iff $X \sqsubseteq a'$.

(15) Let L be a complete lattice and L' be a sublattice of L. If L' is $\bigsqcap$-inheriting, then L' is complete.

(16) Let L be a complete lattice and L' be a sublattice of L. If L' is $\bigsqcup$-inheriting, then L' is complete.

Let L be a complete lattice. Observe that there exists a sublattice of L which is complete.

Let L be a complete lattice. One can check that every sublattice of L which is $\bigsqcap$-inheriting is also complete and every sublattice of L which is $\bigsqcup$-inheriting is also complete.

One can prove the following four propositions:

(17) Let L be a complete lattice and L' be a sublattice of L. Suppose L' is $\bigsqcap$-inheriting. Let A' be a subset of L'. Then $\bigsqcap_L A' = \bigsqcap_{L'} A'$.

(18) Let L be a complete lattice and L' be a sublattice of L. Suppose L' is $\bigsqcup$-inheriting. Let A' be a subset of L'. Then $\bigsqcup_L A' = \bigsqcup_{L'} A'$.

(19) Let L be a complete lattice and L' be a sublattice of L. Suppose L' is $\bigsqcap$-inheriting. Let A be a subset of L and A' be a subset of L'. If $A = A'$, then $\bigsqcap A = \bigsqcap A'$.

(20) Let L be a complete lattice and L' be a sublattice of L. Suppose L' is $\bigsqcup$-inheriting. Let A be a subset of L and A' be a subset of L'. If $A = A'$, then $\bigsqcup A = \bigsqcup A'$.

3. Segment of Real Numbers as a Complete Lattice

Let us consider r_1, r_2. Let us assume that $r_1 \leq r_2$. The functor $\mathrm{RealSubLatt}(r_1, r_2)$ yields a strict lattice and is defined by the conditions (Def. 4).

(Def. 4)(i) The carrier of $\mathrm{RealSubLatt}(r_1, r_2) = [r_1, r_2]$,

(ii) the join operation of $\mathrm{RealSubLatt}(r_1, r_2) = \max_{\mathbb{R}} \restriction ([\!:[r_1, r_2], [r_1, r_2]:\!]$ **qua** set$)$, and

(iii) the meet operation of $\mathrm{RealSubLatt}(r_1, r_2) = \min_{\mathbb{R}} \restriction ([\!:[r_1, r_2], [r_1, r_2]:\!]$ **qua** set$)$.

One can prove the following propositions:

(21) For all r_1, r_2 such that $r_1 \leq r_2$ holds $\mathrm{RealSubLatt}(r_1, r_2)$ is complete.

(22) There exists a sublattice of $\mathrm{RealSubLatt}(0, 1)$ which is $\bigsqcup$-inheriting and non $\bigsqcap$-inheriting.

(23) There exists a complete lattice L such that there exists a sublattice of L which is $\bigsqcup$-inheriting and non $\bigsqcap$-inheriting.

(24) There exists a sublattice of $\mathrm{RealSubLatt}(0, 1)$ which is $\bigsqcap$-inheriting and non $\bigsqcup$-inheriting.

(25) There exists a complete lattice L such that there exists a sublattice of L which is $\bigsqcap$-inheriting and non $\bigsqcup$-inheriting.

References

[1] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/lattice3.html`.

[2] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_2.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[5] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[7] Marek Chmur. The lattice of real numbers. The lattice of real functions. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/real_lat.html`.

[8] Marek Chmur. The lattice of natural numbers and the sublattice of it. The set of prime numbers. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/nat_lat.html`.

[9] Artur Korniłowicz. Certain facts about families of subsets of many sorted sets. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/mssubfam.html`.

[10] Artur Korniłowicz. On the closure operator and the closure system of many sorted sets. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/closure2.html`.

[11] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_3.html`.

[12] Małgorzata Korolkiewicz. Many sorted quotient algebra. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_4.html`.

[13] Beata Madras. Products of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/pralg_2.html`.

[14] Robert Milewski. Lattice of congruences in many sorted algebra. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/msualg_5.html`.

[15] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/setfam_1.html`.

[16] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/eqrel_1.html`.

[17] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rcomp_1.html`.

[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[19] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pboole.html`.

[20] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[21] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

[22] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/lattices.html`.

Received February 9, 1996

Published January 2, 2004