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# Lattice of Congruences in Many Sorted Algebra

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The articles [14], [8], [18], [19], [21], [5], [7], [6], [20], [13], [4], [15], [16], [10], [11], [22], [9], [2], [12], [3], [17], and [1] provide the notation and terminology for this paper.

## 1. More on Equivalence Relations

For simplicity, we adopt the following convention: $I$, $X$ are sets, $M$ is a many sorted set indexed by $I$, $R_1$ is a binary relation on $X$, and $E_1$, $E_2$, $E_3$ are equivalence relations of $X$.

Next we state the proposition

(1) $(E_1 \sqcup E_2) \sqcup E_3 = E_1 \sqcup (E_2 \sqcup E_3)$.

Let $X$ be a set and let $R$ be a binary relation on $X$. The functor $\mathrm{EqCl}(R)$ yields an equivalence relation of $X$ and is defined by:

(Def. 1) $R \subseteq \mathrm{EqCl}(R)$ and for every equivalence relation $E_2$ of $X$ such that $R \subseteq E_2$ holds $\mathrm{EqCl}(R) \subseteq E_2$.

Next we state three propositions:

(2) $E_1 \sqcup E_2 = \mathrm{EqCl}(E_1 \cup E_2)$.

(3) $\mathrm{EqCl}(E_1) = E_1$.

(4) $\nabla_X \cup R_1 = \nabla_X$.

## 2. Lattice of Equivalence Relations

Let $X$ be a set. The functor $\mathrm{EqRelLatt}(X)$ yielding a strict lattice is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of $\mathrm{EqRelLatt}(X) = \{x; x$ ranges over relations between $X$ and $X$: $x$ is an equivalence relation of $X\}$, and

(ii) for all equivalence relations $x$, $y$ of $X$ holds (the meet operation of $\mathrm{EqRelLatt}(X))(x, y) = x \cap y$ and (the join operation of $\mathrm{EqRelLatt}(X))(x, y) = x \sqcup y$.

## 3. Many Sorted Equivalence Relations

Let us consider $I$, $M$. Note that there exists a many sorted relation indexed by $M$ which is equivalence.

Let us consider $I$, $M$. An equivalence relation of $M$ is an equivalence many sorted relation indexed by $M$.

We adopt the following convention: $I$ denotes a non empty set, $M$ denotes a many sorted set indexed by $I$, and $E_4$, $E_1$, $E_2$, $E_3$ denote equivalence relations of $M$.

Let $I$ be a non empty set, let $M$ be a many sorted set indexed by $I$, and let $R$ be a many sorted relation indexed by $M$. The functor $\mathrm{EqCl}(R)$ yields an equivalence relation of $M$ and is defined by:

(Def. 3) For every element $i$ of $I$ holds $(\mathrm{EqCl}(R))(i) = \mathrm{EqCl}(R(i))$.

One can prove the following proposition

(5) $\mathrm{EqCl}(E_4) = E_4$.

## 4. Lattice of Many Sorted Equivalence Relations

Let $I$ be a non empty set, let $M$ be a many sorted set indexed by $I$, and let $E_1$, $E_2$ be equivalence relations of $M$. The functor $E_1 \sqcup E_2$ yielding an equivalence relation of $M$ is defined by:

(Def. 4) There exists a many sorted relation $E_3$ indexed by $M$ such that $E_3 = E_1 \cup E_2$ and $E_1 \sqcup E_2 = \mathrm{EqCl}(E_3)$.

Let us notice that the functor $E_1 \sqcup E_2$ is commutative.

One can prove the following propositions:

(6) $E_1 \cup E_2 \subseteq E_1 \sqcup E_2$.

(7) For every equivalence relation $E_4$ of $M$ such that $E_1 \cup E_2 \subseteq E_4$ holds $E_1 \sqcup E_2 \subseteq E_4$.

(8) If $E_1 \cup E_2 \subseteq E_3$ and for every equivalence relation $E_4$ of $M$ such that $E_1 \cup E_2 \subseteq E_4$ holds $E_3 \subseteq E_4$, then $E_3 = E_1 \sqcup E_2$.

(9) $E_4 \sqcup E_4 = E_4$.

(10) $(E_1 \sqcup E_2) \sqcup E_3 = E_1 \sqcup (E_2 \sqcup E_3)$.

(11) $E_1 \cap (E_1 \sqcup E_2) = E_1$.

(12) For every equivalence relation $E_4$ of $M$ such that $E_4 = E_1 \cap E_2$ holds $E_1 \sqcup E_4 = E_1$.

(13) For all equivalence relations $E_1$, $E_2$ of $M$ holds $E_1 \cap E_2$ is an equivalence relation of $M$.

Let $I$ be a non empty set and let $M$ be a many sorted set indexed by $I$. The functor $\mathrm{EqRelLatt}(M)$ yields a strict lattice and is defined by the conditions (Def. 5).

(Def. 5)(i) For every set $x$ holds $x \in$ the carrier of $\mathrm{EqRelLatt}(M)$ iff $x$ is an equivalence relation of $M$, and

(ii) for all equivalence relations $x$, $y$ of $M$ holds (the meet operation of $\mathrm{EqRelLatt}(M))(x, y) = x \cap y$ and (the join operation of $\mathrm{EqRelLatt}(M))(x, y) = x \sqcup y$.

## 5. Lattice of Congruences in Many Sorted Algebra

Let $S$ be a non empty many sorted signature and let $A$ be an algebra over $S$. Observe that every many sorted relation indexed by $A$ which is equivalence is also equivalence.

In the sequel $S$ denotes a non void non empty many sorted signature and $A$ denotes a non-empty algebra over $S$.

Next we state several propositions:

(14) Let $o$ be an operation symbol of $S$, $C_1$, $C_2$ be congruences of $A$, $x_1$, $y_1$ be sets, and $a_1$, $b_1$ be finite sequences. Suppose $\langle x_1, y_1\rangle \in C_1(\mathrm{Arity}(o)_{\mathrm{len}\, a_1+1}) \cup C_2(\mathrm{Arity}(o)_{\mathrm{len}\, a_1+1})$. Let $x$, $y$ be elements of $\mathrm{Args}(o,A)$. Suppose $x = a_1 \frown \langle x_1\rangle \frown b_1$ and $y = a_1 \frown \langle y_1\rangle \frown b_1$. Then $\langle(\mathrm{Den}(o, A))(x), (\mathrm{Den}(o,A))(y)\rangle \in C_1(\text{the result sort of } o) \cup C_2(\text{the result sort of } o)$.

(15) Let $o$ be an operation symbol of $S$, $C_1$, $C_2$ be congruences of $A$, and $C$ be an equivalence many sorted relation indexed by $A$. Suppose $C = C_1 \sqcup C_2$. Let $x_1$, $y_1$ be sets, $n$ be a natural number, and $a_1$, $a_2$, $b_1$ be finite sequences. Suppose $\mathrm{len}\, a_1 = n$ and $\mathrm{len}\, a_1 = \mathrm{len}\, a_2$ and for every natural number $k$ such that $k \in \mathrm{dom}\, a_1$ holds $\langle a_1(k), a_2(k)\rangle \in C(\mathrm{Arity}(o)_k)$. Suppose $\langle(\mathrm{Den}(o,A))(a_1 \frown \langle x_1\rangle \frown b_1), (\mathrm{Den}(o,A))(a_2 \frown \langle x_1\rangle \frown b_1)\rangle \in C(\text{the result sort of } o)$ and $\langle x_1, y_1\rangle \in C(\mathrm{Arity}(o)_{n+1})$. Let $x$ be an element of $\mathrm{Args}(o,A)$. If $x = a_1 \frown \langle x_1\rangle \frown b_1$, then $\langle(\mathrm{Den}(o, A))(x), (\mathrm{Den}(o,A))(a_2 \frown \langle y_1\rangle \frown b_1)\rangle \in C(\text{the result sort of } o)$.

(16) Let $o$ be an operation symbol of $S$, $C_1$, $C_2$ be congruences of $A$, and $C$ be an equivalence many sorted relation indexed by $A$. Suppose $C = C_1 \sqcup C_2$. Let $x$, $y$ be elements of $\mathrm{Args}(o,A)$. Suppose that for every natural number $n$ such that $n \in \mathrm{dom}\, x$ holds $\langle x(n), y(n)\rangle \in C(\mathrm{Arity}(o)_n)$. Then $\langle(\mathrm{Den}(o,A))(x), (\mathrm{Den}(o,A))(y)\rangle \in C(\text{the result sort of } o)$.

(17) For all congruences $C_1$, $C_2$ of $A$ holds $C_1 \sqcup C_2$ is a congruence of $A$.

(18) For all congruences $C_1$, $C_2$ of $A$ holds $C_1 \cap C_2$ is a congruence of $A$.

Let us consider $S$ and let $A$ be a non-empty algebra over $S$. The functor $\mathrm{CongrLatt}(A)$ yields a strict sublattice of $\mathrm{EqRelLatt}(\text{the sorts of } A)$ and is defined as follows:

(Def. 6) For every set $x$ holds $x \in$ the carrier of $\mathrm{CongrLatt}(A)$ iff $x$ is a congruence of $A$.

We now state two propositions:

(19) $\mathrm{id}_{\text{the sorts of } A}$ is a congruence of $A$.

(20) $[\![\text{the sorts of } A, \text{the sorts of } A]\!]$ is a congruence of $A$.

Let us consider $S$, $A$. Observe that $\mathrm{CongrLatt}(A)$ is bounded.

We now state two propositions:

(21) $\bot_{\mathrm{CongrLatt}(A)} = \mathrm{id}_{\text{the sorts of } A}$.

(22) $\top_{\mathrm{CongrLatt}(A)} = [\![\text{the sorts of } A, \text{the sorts of } A]\!]$.

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