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Homomorphisms of Many Sorted Algebras

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Summary. The aim of this article is to present the definition and some properties of homomorphisms of many sorted algebras. Some auxiliary properties of many sorted functions also have been shown.

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The articles [11], [15], [16], [6], [8], [7], [4], [2], [14], [1], [3], [12], [9], [13], [5], and [10] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we use the following convention: S is a non void non empty many sorted signature, U_1, U_2 are algebras over S, o is an operation symbol of S, and n is a natural number.

Let I be a non empty set, let A, B be many sorted sets indexed by I, let F be a many sorted function from A into B, and let i be an element of I. Then $F(i)$ is a function from $A(i)$ into $B(i)$.

Let us consider S and let U_1, U_2 be algebras over S. A many sorted function from U_1 into U_2 is a many sorted function from the sorts of U_1 into the sorts of U_2.

Let I be a set and let A be a many sorted set indexed by I. The functor id_A yields a many sorted function from A into A and is defined as follows:

(Def. 1) For every set i such that $i \in I$ holds $\mathrm{id}_A(i) = \mathrm{id}_{A(i)}$.

Let I_1 be a function. We say that I_1 is "1-1" if and only if:

(Def. 2) For every set i and for every function f such that $i \in \mathrm{dom}\, I_1$ and $I_1(i) = f$ holds f is one-to-one.

Let I be a set. Observe that there exists a many sorted function indexed by I which is "1-1".

Next we state the proposition

(1) Let I be a set and F be a many sorted function indexed by I. Then F is "1-1" if and only if for every set i such that $i \in I$ holds $F(i)$ is one-to-one.

Let I be a set, let A, B be many sorted sets indexed by I, and let I_1 be a many sorted function from A into B. We say that I_1 is onto if and only if:

(Def. 3) For every set i such that $i \in I$ holds $\mathrm{rng}\, I_1(i) = B(i)$.

Let F, G be function yielding functions. The functor $G \circ F$ yields a function yielding function and is defined as follows:

(Def. 4) $\mathrm{dom}(G \circ F) = \mathrm{dom}\, F \cap \mathrm{dom}\, G$ and for every set i such that $i \in \mathrm{dom}(G \circ F)$ holds $(G \circ F)(i) = G(i) \cdot F(i)$.

One can prove the following proposition

(2) Let I be a set, A, B, C be many sorted sets indexed by I, F be a many sorted function from A into B, and G be a many sorted function from B into C. Then $\mathrm{dom}(G \circ F) = I$ and for every set i such that $i \in I$ holds $(G \circ F)(i) = G(i) \cdot F(i)$.

Let I be a set, let A be a many sorted set indexed by I, let B, C be non-empty many sorted sets indexed by I, let F be a many sorted function from A into B, and let G be a many sorted function from B into C. Then $G \circ F$ is a many sorted function from A into C.

One can prove the following propositions:

(3) Let I be a set, A, B be many sorted sets indexed by I, and F be a many sorted function from A into B. Then $F \circ \mathrm{id}_A = F$.

(4) Let I be a set, A, B be many sorted sets indexed by I, and F be a many sorted function from A into B. Then $\mathrm{id}_B \circ F = F$.

Let I be a set, let A, B be many sorted sets indexed by I, and let F be a many sorted function from A into B. Let us assume that F is "1-1" and onto. The functor F^{-1} yielding a many sorted function from B into A is defined by:

(Def. 5) For every set i such that $i \in I$ holds $F^{-1}(i) = F(i)^{-1}$.

Next we state the proposition

(5) Let I be a set, A, B be non-empty many sorted sets indexed by I, H be a many sorted function from A into B, and H_1 be a many sorted function from B into A. If H is "1-1" and onto and $H_1 = H^{-1}$, then $H \circ H_1 = \mathrm{id}_B$ and $H_1 \circ H = \mathrm{id}_A$.

Let I be a set, let A be a many sorted set indexed by I, and let F be a many sorted function indexed by I. The functor $F^\circ A$ yields a many sorted set indexed by I and is defined as follows:

(Def. 6) For every set i such that $i \in I$ holds $(F^\circ A)(i) = F(i)^\circ A(i)$.

Let us consider S, let U_1 be a non-empty algebra over S, and let us consider o. One can check that every element of $\mathrm{Args}(o, U_1)$ is function-like and relation-like.

2. Homomorphisms of Many Sorted Algebras

We now state the proposition

(6) Let U_1 be an algebra over S and x be a function. Suppose $x \in \mathrm{Args}(o, U_1)$. Then $\mathrm{dom}\, x = \mathrm{dom}\, \mathrm{Arity}(o)$ and for every set y such that $y \in \mathrm{dom}((\text{the sorts of } U_1) \cdot \mathrm{Arity}(o))$ holds $x(y) \in ((\text{the sorts of } U_1) \cdot \mathrm{Arity}(o))(y)$.

Let us consider S, let U_1, U_2 be algebras over S, let us consider o, let F be a many sorted function from U_1 into U_2, and let x be an element of $\mathrm{Args}(o, U_1)$. Let us assume that $\mathrm{Args}(o, U_1) \neq \emptyset$ and $\mathrm{Args}(o, U_2) \neq \emptyset$. The functor $F\#x$ yields an element of $\mathrm{Args}(o, U_2)$ and is defined as follows:

(Def. 7) $F\#x = (\mathrm{Frege}(F \cdot \mathrm{Arity}(o)))(x)$.

Let us consider S, let U_1 be a non-empty algebra over S, and let us consider o. One can verify that there exists an element of $\mathrm{Args}(o, U_1)$ which is function-like and relation-like.

Let us consider S, let U_1, U_2 be non-empty algebras over S, let us consider o, let F be a many sorted function from U_1 into U_2, and let x be an element of $\mathrm{Args}(o, U_1)$. Then $F\#x$ is a function-like relation-like element of $\mathrm{Args}(o, U_2)$ and it can be characterized by the condition:

(Def. 8) For every n such that $n \in \mathrm{dom}\, x$ holds $(F\#x)(n) = F(\mathrm{Arity}(o)_n)(x(n))$.

We now state two propositions:

(7) Let given S, o and U_1 be an algebra over S. If $\mathrm{Args}(o, U_1) \neq \emptyset$, then for every element x of $\mathrm{Args}(o, U_1)$ holds $x = \mathrm{id}_{\text{the sorts of } U_1} \# x$.

(8) Let U_1, U_2, U_3 be non-empty algebras over S, H_1 be a many sorted function from U_1 into U_2, H_2 be a many sorted function from U_2 into U_3, and x be an element of $\mathrm{Args}(o, U_1)$. Then $(H_2 \circ H_1) \# x = H_2 \# (H_1 \# x)$.

Let us consider S, let U_1, U_2 be algebras over S, and let F be a many sorted function from U_1 into U_2. We say that F is a homomorphism of U_1 into U_2 if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let o be an operation symbol of S. Suppose $\mathrm{Args}(o, U_1) \neq \emptyset$. Let x be an element of $\mathrm{Args}(o, U_1)$. Then $F(\text{the result sort of } o)((\mathrm{Den}(o, U_1))(x)) = (\mathrm{Den}(o, U_2))(F \# x)$.

Next we state two propositions:

(9) For every algebra U_1 over S holds $\mathrm{id}_{\text{the sorts of } U_1}$ is a homomorphism of U_1 into U_1.

(10) Let U_1, U_2, U_3 be non-empty algebras over S, H_1 be a many sorted function from U_1 into U_2, and H_2 be a many sorted function from U_2 into U_3. Suppose H_1 is a homomorphism of U_1 into U_2 and H_2 is a homomorphism of U_2 into U_3. Then $H_2 \circ H_1$ is a homomorphism of U_1 into U_3.

Let us consider S, let U_1, U_2 be algebras over S, and let F be a many sorted function from U_1 into U_2. We say that F is an epimorphism of U_1 onto U_2 if and only if:

(Def. 10) F is a homomorphism of U_1 into U_2 and onto.

One can prove the following proposition

(11) Let U_1, U_2, U_3 be non-empty algebras over S, F be a many sorted function from U_1 into U_2, and G be a many sorted function from U_2 into U_3. Suppose F is an epimorphism of U_1 onto U_2 and G is an epimorphism of U_2 onto U_3. Then $G \circ F$ is an epimorphism of U_1 onto U_3.

Let us consider S, let U_1, U_2 be algebras over S, and let F be a many sorted function from U_1 into U_2. We say that F is a monomorphism of U_1 into U_2 if and only if:

(Def. 11) F is a homomorphism of U_1 into U_2 and "1-1".

The following proposition is true

(12) Let U_1, U_2, U_3 be non-empty algebras over S, F be a many sorted function from U_1 into U_2, and G be a many sorted function from U_2 into U_3. Suppose F is a monomorphism of U_1 into U_2 and G is a monomorphism of U_2 into U_3. Then $G \circ F$ is a monomorphism of U_1 into U_3.

Let us consider S, let U_1, U_2 be algebras over S, and let F be a many sorted function from U_1 into U_2. We say that F is an isomorphism of U_1 and U_2 if and only if:

(Def. 12) F is an epimorphism of U_1 onto U_2 and a monomorphism of U_1 into U_2.

One can prove the following three propositions:

(13) Let F be a many sorted function from U_1 into U_2. Then F is an isomorphism of U_1 and U_2 if and only if F is a homomorphism of U_1 into U_2, onto, and "1-1".

(14) Let U_1, U_2 be non-empty algebras over S, H be a many sorted function from U_1 into U_2, and H_1 be a many sorted function from U_2 into U_1. Suppose H is an isomorphism of U_1 and U_2 and $H_1 = H^{-1}$. Then H_1 is an isomorphism of U_2 and U_1.

(15) Let U_1, U_2, U_3 be non-empty algebras over S, H be a many sorted function from U_1 into U_2, and H_1 be a many sorted function from U_2 into U_3. Suppose H is an isomorphism of U_1 and U_2 and H_1 is an isomorphism of U_2 and U_3. Then $H_1 \circ H$ is an isomorphism of U_1 and U_3.

Let us consider S and let U_1, U_2 be algebras over S. We say that U_1 and U_2 are isomorphic if and only if:

(Def. 13) There exists a many sorted function from U_1 into U_2 which is an isomorphism of U_1 and U_2.

Next we state the proposition

(16) For every algebra U_1 over S holds $\mathrm{id}_{\text{the sorts of } U_1}$ is an isomorphism of U_1 and U_1 and U_1 and U_1 are isomorphic.

Let us consider S and let U_1, U_2 be algebras over S. Let us note that the predicate U_1 and U_2 are isomorphic is reflexive.

Next we state two propositions:

(17) For all non-empty algebras U_1, U_2 over S such that U_1 and U_2 are isomorphic holds U_2 and U_1 are isomorphic.

(18) Let U_1, U_2, U_3 be non-empty algebras over S. Suppose U_1 and U_2 are isomorphic and U_2 and U_3 are isomorphic. Then U_1 and U_3 are isomorphic.

Let us consider S, let U_1, U_2 be non-empty algebras over S, and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2. The functor $\mathrm{Im} F$ yields a strict non-empty subalgebra of U_2 and is defined as follows:

(Def. 14) The sorts of $\mathrm{Im} F = F^{\circ}$(the sorts of U_1).

Next we state several propositions:

(19) Let U_1 be a non-empty algebra over S, U_2 be a strict non-empty algebra over S, and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then F is an epimorphism of U_1 onto U_2 if and only if $\mathrm{Im} F = U_2$.

(20) Let U_1, U_2 be non-empty algebras over S, F be a many sorted function from U_1 into U_2, and G be a many sorted function from U_1 into $\mathrm{Im} F$. Suppose $F = G$ and F is a homomorphism of U_1 into U_2. Then G is an epimorphism of U_1 onto $\mathrm{Im} F$.

(21) Let U_1, U_2 be non-empty algebras over S and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then there exists a many sorted function G from U_1 into $\mathrm{Im} F$ such that $F = G$ and G is an epimorphism of U_1 onto $\mathrm{Im} F$.

(22) Let U_1 be a non-empty algebra over S, U_2 be a non-empty subalgebra of U_1, and G be a many sorted function from U_2 into U_1. If $G = \mathrm{id}_{\text{the sorts of } U_2}$, then G is a monomorphism of U_2 into U_1.

(23) Let U_1, U_2 be non-empty algebras over S and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then there exists a many sorted function F_1 from U_1 into $\mathrm{Im} F$ and there exists a many sorted function F_2 from $\mathrm{Im} F$ into U_2 such that F_1 is an epimorphism of U_1 onto $\mathrm{Im} F$ and F_2 is a monomorphism of $\mathrm{Im} F$ into U_2 and $F = F_2 \circ F_1$.

(24) Let given S, U_1, U_2 be algebras over S, given o, F be a many sorted function from U_1 into U_2, x be an element of $\mathrm{Args}(o, U_1)$, and f, u be functions. Suppose $x = f$ and $x \in \mathrm{Args}(o, U_1)$ and $u \in \mathrm{Args}(o, U_2)$. Then $u = F \# x$ if and only if for every n such that $n \in \mathrm{dom} f$ holds $u(n) = F(\mathrm{Arity}(o)_n)(f(n))$.

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