The Correspondence Between Lattices of Subalgebras of Universal Algebras and Many Sorted Algebras

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Summary. The main goal of this paper is to show some properties of subalgebras of universal algebras and many sorted algebras, and then the isomorphic correspondence between lattices of such subalgebras.

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The articles [11], [15], [8], [17], [16], [5], [7], [6], [12], [10], [3], [2], [14], [1], [9], [13], and [4] provide the notation and terminology for this paper.

1. PRELIMINARIES

In this paper a is a set and i is a natural number.

The following propositions are true:

(1) $(\square \longmapsto a)(0) = \emptyset$.

(2) $(\square \longmapsto a)(1) = \langle a \rangle$.

(3) $(\square \longmapsto a)(2) = \langle a, a \rangle$.

(4) $(\square \longmapsto a)(3) = \langle a, a, a \rangle$.

(5) For every finite sequence f of elements of $\{0\}$ holds $f = i \mapsto 0$ iff $\operatorname{len} f = i$.

(6) For every finite sequence f such that $f = (\square \longmapsto 0)(i)$ holds $\operatorname{len} f = i$.

2. SOME PROPERTIES OF SUBALGEBRAS OF UNIVERSAL AND MANY SORTED ALGEBRAS

Next we state the proposition

(7) For all universal algebras U_1, U_2 such that U_1 is a subalgebra of U_2 holds $\operatorname{MSSign}(U_1) = \operatorname{MSSign}(U_2)$.

Let U_0 be a universal algebra. One can check that the characteristic of U_0 is function yielding. One can prove the following propositions:

(8)   Let U_1, U_2 be universal algebras. Suppose U_1 is a subalgebra of U_2. Let B be a sub-set of MSAlg(U_2). Suppose B = the sorts of MSAlg(U_1). Let o be an operation symbol of MSSign(U_2) and a be an operation symbol of MSSign(U_1). If $a = o$, then Den(a, MSAlg(U_1)) = Den(o, MSAlg(U_2))↾Args(a, MSAlg(U_1)).

(9)   For all universal algebras U_1, U_2 such that U_1 is a subalgebra of U_2 holds the sorts of MSAlg(U_1) are a subset of MSAlg(U_2).

(10)   Let U_1, U_2 be universal algebras. Suppose U_1 is a subalgebra of U_2. Let B be a subset of MSAlg(U_2). If B = the sorts of MSAlg(U_1), then B is operations closed.

(11)   Let U_1, U_2 be universal algebras. Suppose U_1 is a subalgebra of U_2. Let B be a subset of MSAlg(U_2). If B = the sorts of MSAlg(U_1), then the characteristics of MSAlg(U_1) = Opers(MSAlg(U_2), B).

(12)   For all universal algebras U_1, U_2 such that U_1 is a subalgebra of U_2 holds MSAlg(U_1) is a subalgebra of MSAlg(U_2).

(13)   For all universal algebras U_1, U_2 such that MSAlg(U_1) is a subalgebra of MSAlg(U_2) holds the carrier of U_1 is a subset of U_2.

(14)   Let U_1, U_2 be universal algebras. Suppose MSAlg(U_1) is a subalgebra of MSAlg(U_2). Let B be a non empty subset of U_2. If B = the carrier of U_1, then B is operations closed.

(15)   Let U_1, U_2 be universal algebras. Suppose MSAlg(U_1) is a subalgebra of MSAlg(U_2). Let B be a non empty subset of U_2. If B = the carrier of U_1, then the characteristic of U_1 = Opers(U_2, B).

(16)   For all universal algebras U_1, U_2 such that MSAlg(U_1) is a subalgebra of MSAlg(U_2) holds U_1 is a subalgebra of U_2.

In the sequel M_1 is a segmental trivial non void non empty many sorted signature and A is a non-empty algebra over M_1.

The following propositions are true:

(17)   For every non-empty subalgebra B of A holds the carrier of $\mathrm{Alg_1}(B)$ is a subset of $\mathrm{Alg_1}(A)$.

(18)   Let B be a non-empty subalgebra of A and S be a non empty subset of $\mathrm{Alg_1}(A)$. If S = the carrier of $\mathrm{Alg_1}(B)$, then S is operations closed.

(19)   Let B be a non-empty subalgebra of A and S be a non empty subset of $\mathrm{Alg_1}(A)$. If S = the carrier of $\mathrm{Alg_1}(B)$, then the characteristic of $\mathrm{Alg_1}(B)$ = Opers($\mathrm{Alg_1}(A)$, S).

(20)   For every non-empty subalgebra B of A holds $\mathrm{Alg_1}(B)$ is a subalgebra of $\mathrm{Alg_1}(A)$.

(21)   Let S be a non empty non void many sorted signature and A, B be algebras over S. Then A is a subalgebra of B if and only if A is a subalgebra of the algebra of B.

(22)   For all universal algebras A, B holds signature$A$ = signature$B$ iff MSSign(A) = MSSign(B).

(23)   Let A be a non-empty algebra over M_1. Suppose the carrier of M_1 = {0}. Then MSSign($\mathrm{Alg_1}(A)$) = the many sorted signature of M_1.

(24)   Let A, B be non-empty algebras over M_1. Suppose the carrier of M_1 = {0} and $\mathrm{Alg_1}(A)$ = $\mathrm{Alg_1}(B)$. Then the algebra of A = the algebra of B.

(25)   Let A be a non-empty algebra over M_1. If the carrier of M_1 = {0}, then the sorts of A = the sorts of MSAlg($\mathrm{Alg_1}(A)$).

(26)   For every non-empty algebra A over M_1 such that the carrier of M_1 = {0} holds MSAlg($\mathrm{Alg_1}(A)$) = the algebra of A.

(27)   Let A be a universal algebra and B be a strict non-empty subalgebra of MSAlg(A). If the carrier of MSSign(A) = {0}, then $\mathrm{Alg_1}(B)$ is a subalgebra of A.

3. THE CORRESPONDENCE BETWEEN LATTICES OF SUBALGEBRAS OF UNIVERSAL AND MANY SORTED ALGEBRAS

Next we state three propositions:

(28) Let A be a universal algebra, a_1, b_1 be strict non-empty subalgebras of A, and a_2, b_2 be strict non-empty subalgebras of MSAlg(A). Suppose $a_2 = $ MSAlg(a_1) and $b_2 = $ MSAlg(b_1). Then (the sorts of a_2) $\cup$ (the sorts of b_2) $= 0 \longmapsto ((\text{the carrier of } a_1) \cup (\text{the carrier of } b_1))$.

(29) Let A be a universal algebra, a_1, b_1 be strict non-empty subalgebras of A, and a_2, b_2 be strict non-empty subalgebras of MSAlg(A). Suppose $a_2 = $ MSAlg(a_1) and $b_2 = $ MSAlg(b_1). Then (the sorts of a_2) $\cap$ (the sorts of b_2) $= 0 \longmapsto (\text{the carrier of } a_1) \cap (\text{the carrier of } b_1)$.

(30) Let A be a strict universal algebra, a_1, b_1 be strict non-empty subalgebras of A, and a_2, b_2 be strict non-empty subalgebras of MSAlg(A). If $a_2 = $ MSAlg(a_1) and $b_2 = $ MSAlg(b_1), then MSAlg($a_1 \sqcup b_1$) $= a_2 \sqcup b_2$.

Let A be a universal algebra with constants. One can verify that MSSign(A) is non void, strict, segmental, and trivial and has constant operations.

The following proposition is true

(31) Let A be a universal algebra with constants, a_1, b_1 be strict non-empty subalgebras of A, and a_2, b_2 be strict non-empty subalgebras of MSAlg(A). If $a_2 = $ MSAlg(a_1) and $b_2 = $ MSAlg(b_1), then MSAlg($a_1 \cap b_1$) $= a_2 \cap b_2$.

Let A be a quasi total universal algebra structure. Note that the universal algebra structure of A is quasi total.

Let A be a partial universal algebra structure. Note that the universal algebra structure of A is partial.

Let A be a non-empty universal algebra structure. Observe that the universal algebra structure of A is non-empty.

Let A be a universal algebra with constants. Note that the universal algebra structure of A has constants.

We now state the proposition

(32) Let A be a universal algebra with constants. Then the lattice of subalgebras of the universal algebra structure of A and the lattice of subalgebras of MSAlg(the universal algebra structure of A) are isomorphic.

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