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A Scheme for Extensions of Homomorphisms of Many Sorted Algebras

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Summary. The aim of this work is to provide a bridge between the theory of context-free grammars developed in [10], [6] and universally free manysorted algebras([14]. The third scheme proved in the article allows to prove that two homomorphisms equal on the set of free generators are equal. The first scheme is a slight modification of the scheme in [6] and the second is rather technical, but since it was useful for me, perhaps it might be useful for somebody else. The concept of flattening of a many sorted function F between two manysorted sets A and B (with common set of indices I) is introduced for A with mutually disjoint components (pairwise disjoint function – the concept introduced in [13]). This is a function on the union of A, that is equal to F on every component of A. A trivial many sorted algebra over a signature S is defined with sorts being singletons of corresponding sort symbols. It has mutually disjoint sorts.

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The articles [15], [9], [18], [19], [7], [8], [5], [13], [1], [2], [3], [4], [10], [6], [12], [16], [17], [14], and [11] provide the notation and terminology for this paper.

The following proposition is true

(1) For all functions f, g such that $g \in \prod f$ holds $\mathrm{rng}\, g \subseteq \bigcup f$.

The scheme *DTConstrUniq* deals with a non empty tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and functions $\mathcal{C}$, $\mathcal{D}$ from $\mathrm{TS}(\mathcal{A})$ into $\mathcal{B}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters meet the following conditions:

- For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{C}$(the root tree of $t) = \mathcal{F}(t)$,
- Let n_1 be a symbol of $\mathcal{A}$ and t_1 be a finite sequence of elements of $\mathrm{TS}(\mathcal{A})$. Suppose $n_1 \Rightarrow$ the roots of t_1. Let x be a finite sequence of elements of $\mathcal{B}$. If $x = \mathcal{C} \cdot t_1$, then $\mathcal{C}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, t_1, x)$,
- For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{D}$(the root tree of $t) = \mathcal{F}(t)$, and
- Let n_1 be a symbol of $\mathcal{A}$ and t_1 be a finite sequence of elements of $\mathrm{TS}(\mathcal{A})$. Suppose $n_1 \Rightarrow$ the roots of t_1. Let x be a finite sequence of elements of $\mathcal{B}$. If $x = \mathcal{D} \cdot t_1$, then $\mathcal{D}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, t_1, x)$.

Next we state two propositions:

(2) Let S be a non void non empty many sorted signature, X be a many sorted set indexed by the carrier of S, and o, b be sets. Suppose $\langle o, b\rangle \in \mathrm{REL}(X)$. Then

(i) $o \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}:\!]$, and

(ii) $b \in ([\!:$ the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X))^{*}$.

(3) Let S be a non void non empty many sorted signature, X be a many sorted set indexed by the carrier of S, o be an operation symbol of S, and b be a finite sequence. Suppose $\langle\langle o$, the carrier of $S\rangle, b\rangle \in \mathrm{REL}(X)$. Then

(i) $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$, and

(ii) for every set x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 = \mathrm{Arity}(o)(x)$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then $b(x) \in \mathrm{coprod}(\mathrm{Arity}(o)(x), X)$.

Let D be a set. We see that the finite sequence of elements of D is an element of D^{*}.

Let I be a non empty set and let M be a non-empty many sorted set indexed by I. One can verify that $\mathrm{rng}\, M$ is non empty and has non empty elements.

Let D be a non empty set with non empty elements. One can verify that $\bigcup D$ is non empty.

Let I be a set. Note that every many sorted set indexed by I which is empty yielding is also disjoint valued.

Let I be a set. Observe that there exists a many sorted set indexed by I which is disjoint valued.

Let I be a non empty set, let X be a disjoint valued many sorted set indexed by I, let D be a non-empty many sorted set indexed by I, and let F be a many sorted function from X into D. The functor $\mathrm{Flatten}(F)$ yielding a function from $\bigcup X$ into $\bigcup D$ is defined as follows:

(Def. 1) For every element i of I and for every set x such that $x \in X(i)$ holds $(\mathrm{Flatten}(F))(x) = F(i)(x)$.

Next we state the proposition

(4) Let I be a non empty set, X be a disjoint valued many sorted set indexed by I, D be a non-empty many sorted set indexed by I, and F_1, F_2 be many sorted functions from X into D. If $\mathrm{Flatten}(F_1) = \mathrm{Flatten}(F_2)$, then $F_1 = F_2$.

Let S be a non empty many sorted signature and let A be an algebra over S. We say that A is disjoint valued if and only if:

(Def. 2) The sorts of A are disjoint valued.

Let S be a non empty many sorted signature. The functor $\mathrm{SingleAlg}(S)$ yielding a strict algebra over S is defined by:

(Def. 3) For every set i such that $i \in$ the carrier of S holds (the sorts of $\mathrm{SingleAlg}(S))(i) = \{i\}$.

Let S be a non empty many sorted signature. One can verify that there exists an algebra over S which is non-empty and disjoint valued.

Let S be a non empty many sorted signature. One can verify that $\mathrm{SingleAlg}(S)$ is non-empty and disjoint valued.

Let S be a non empty many sorted signature and let A be a disjoint valued algebra over S. Note that the sorts of A is disjoint valued.

Next we state the proposition

(5) Let S be a non void non empty many sorted signature, o be an operation symbol of S, A_1 be a non-empty disjoint valued algebra over S, A_2 be a non-empty algebra over S, f be a many sorted function from A_1 into A_2, and a be an element of $\mathrm{Args}(o, A_1)$. Then $\mathrm{Flatten}(f) \cdot a = f\# a$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Observe that $\mathrm{FreeSorts}(X)$ is disjoint valued.

The scheme *FreeSortUniq* deals with a non void non empty many sorted signature $\mathcal{A}$, non-empty many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by the carrier of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\bigcup \mathcal{C}$, a ternary functor $\mathcal{G}$ yielding an element of $\bigcup \mathcal{C}$, and many sorted functions $\mathcal{D}$, $\mathcal{E}$ from $\mathrm{FreeSorts}(\mathcal{B})$ into $\mathcal{C}$, and states that:

$\mathcal{D} = \mathcal{E}$

provided the following conditions are satisfied:

- Let o be an operation symbol of $\mathcal{A}$, t_1 be an element of $\mathrm{Args}(o, \mathrm{Free}(\mathcal{B}))$, and x be a finite sequence of elements of $\bigcup \mathcal{C}$. If $x = \mathrm{Flatten}(\mathcal{D}) \cdot t_1$, then $\mathcal{D}$(the result sort of o)$((\mathrm{Den}(o, \mathrm{Free}(\mathcal{B})))(t_1)) = \mathcal{G}(o, t_1, x)$,
- For every sort symbol s of $\mathcal{A}$ and for every set y such that $y \in \mathrm{FreeGenerator}(s, \mathcal{B})$ holds $\mathcal{D}(s)(y) = \mathcal{F}(y)$,
- Let o be an operation symbol of $\mathcal{A}$, t_1 be an element of $\mathrm{Args}(o, \mathrm{Free}(\mathcal{B}))$, and x be a finite sequence of elements of $\bigcup \mathcal{C}$. If $x = \mathrm{Flatten}(\mathcal{E}) \cdot t_1$, then $\mathcal{E}$(the result sort of o)$((\mathrm{Den}(o, \mathrm{Free}(\mathcal{B})))(t_1)) = \mathcal{G}(o, t_1, x)$, and
- For every sort symbol s of $\mathcal{A}$ and for every set y such that $y \in \mathrm{FreeGenerator}(s, \mathcal{B})$ holds $\mathcal{E}(s)(y) = \mathcal{F}(y)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Observe that $\mathrm{Free}(X)$ is non-empty.

Let S be a non void non empty many sorted signature, let o be an operation symbol of S, and let A be a non-empty algebra over S. One can check that $\mathrm{Args}(o, A)$ is non empty and $\mathrm{Result}(o, A)$ is non empty.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Note that the sorts of $\mathrm{Free}(X)$ is disjoint valued.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Observe that $\mathrm{Free}(X)$ is disjoint valued.

The scheme *ExtFreeGen* deals with a non void non empty many sorted signature $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by the carrier of $\mathcal{A}$, a non-empty algebra $\mathcal{C}$ over $\mathcal{A}$, many sorted functions $\mathcal{D}$, $\mathcal{E}$ from $\mathrm{Free}(\mathcal{B})$ into $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

$\mathcal{D} = \mathcal{E}$

provided the following requirements are met:

- $\mathcal{D}$ is a homomorphism of $\mathrm{Free}(\mathcal{B})$ into $\mathcal{C}$,
- For every sort symbol s of $\mathcal{A}$ and for all sets x, y such that $y \in \mathrm{FreeGenerator}(s, \mathcal{B})$ holds $\mathcal{D}(s)(y) = x$ iff $\mathcal{P}[s, x, y]$,
- $\mathcal{E}$ is a homomorphism of $\mathrm{Free}(\mathcal{B})$ into $\mathcal{C}$, and
- For every sort symbol s of $\mathcal{A}$ and for all sets x, y such that $y \in \mathrm{FreeGenerator}(s, \mathcal{B})$ holds $\mathcal{E}(s)(y) = x$ iff $\mathcal{P}[s, x, y]$.

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