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Monoid of Multisets and Subsets

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Summary. The monoid of functions yielding elements of a group is introduced. The monoid of multisets over a set is constructed as such monoid where the target group is the group of natural numbers with addition. Moreover, the generalization of group operation onto the operation on subsets is present. That generalization is used to introduce the group 2^G of subsets of a group G.

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The articles [17], [10], [21], [20], [2], [22], [8], [5], [4], [9], [7], [14], [16], [12], [19], [6], [11], [1], [18], [3], [13], and [15] provide the notation and terminology for this paper.

1. Updating

We use the following convention: x, y, X, Y, Z denote sets and n denotes a natural number.

Let D_1, D_2, D be non empty sets. A binary function from D_1, D_2 into D is a function from $[: D_1, D_2 :]$ into D.

Let f be a function and let x_1, x_2, y be sets. The functor $f(x_1,x_2)(y)$ is defined by:

(Def. 1) $f(x_1,x_2)(y) = f(\langle x_1, x_2 \rangle)(y)$.

The following proposition is true

(1) For all functions f, g and for all sets x_1, x_2, x such that $\langle x_1, x_2 \rangle \in \mathrm{dom}\, f$ and $g = f(x_1, x_2)$ and $x \in \mathrm{dom}\, g$ holds $f(x_1,x_2)(x) = g(x)$.

Let A, D_1, D_2, D be non empty sets, let f be a binary function from D_1, D_2 into D^A, let x_1 be an element of D_1, let x_2 be an element of D_2, and let x be an element of A. Then $f(x_1,x_2)(x)$ is an element of D.

Let A be a set, let D_1, D_2, D be non empty sets, let f be a binary function from D_1, D_2 into D, let g_1 be a function from A into D_1, and let g_2 be a function from A into D_2. Then $f^\circ(g_1, g_2)$ is an element of D^A.

Let A be a non empty set, let n be a natural number, and let x be an element of A. Then $n \mapsto x$ is a finite sequence of elements of A. We introduce $n \dot{\longmapsto} x$ as a synonym of $n \mapsto x$.

Let D be a non empty set, let A be a set, and let d be an element of D. Then $A \longmapsto d$ is an element of D^A.

Let A be a set, let D_1, D_2, D be non empty sets, let f be a binary function from D_1, D_2 into D, let d be an element of D_1, and let g be a function from A into D_2. Then $f^\circ(d,g)$ is an element of D^A.

Let A be a set, let D_1, D_2, D be non empty sets, let f be a binary function from D_1, D_2 into D, let g be a function from A into D_1, and let d be an element of D_2. Then $f^\circ(g,d)$ is an element of D^A.

One can prove the following proposition

(2) For all functions f, g and for every set X holds $(f{\restriction}X)\cdot g = f\cdot(X{\upharpoonleft}g)$.

The scheme *NonUniqFuncDEx* deals with a set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters satisfy the following condition:

- For every x such that $x \in \mathcal{A}$ there exists an element y of $\mathcal{B}$ such that $\mathcal{P}[x,y]$.

2. Monoid of Functions into a Semigroup

Let D_1, D_2, D be non empty sets, let f be a binary function from D_1, D_2 into D, and let A be a set. The functor f°_A yielding a binary function from ${D_1}^A$, ${D_2}^A$ into D^A is defined as follows:

(Def. 2) For every element f_1 of ${D_1}^A$ and for every element f_2 of ${D_2}^A$ holds $(f^\circ_A)(f_1, f_2) = f^\circ(f_1, f_2)$.

Next we state the proposition

(3) Let D_1, D_2, D be non empty sets, f be a binary function from D_1, D_2 into D, A be a set, f_1 be a function from A into D_1, f_2 be a function from A into D_2, and given x. If $x \in A$, then $(f^\circ_A)(f_1, f_2)(x) = f(f_1(x), f_2(x))$.

For simplicity, we adopt the following convention: A is a set, D is a non empty set, a is an element of D, o, o' are binary operations on D, and f, g, h are functions from A into D.

The following propositions are true:

(4) If o is commutative, then $o^\circ(f,g) = o^\circ(g, f)$.

(5) If o is associative, then $o^\circ(o^\circ(f,g),h) = o^\circ(f, o^\circ(g,h))$.

(6) If a is a unity w.r.t. o, then $o^\circ(a, f) = f$ and $o^\circ(f,a) = f$.

(7) If o is idempotent, then $o^\circ(f, f) = f$.

(8) If o is commutative, then o°_A is commutative.

(9) If o is associative, then o°_A is associative.

(10) If a is a unity w.r.t. o, then $A \longmapsto a$ is a unity w.r.t. o°_A.

(11) If o has a unity, then $\mathbf{1}_{o^\circ_A} = A \longmapsto \mathbf{1}_o$ and o°_A has a unity.

(12) If o is idempotent, then o°_A is idempotent.

(13) If o is invertible, then o°_A is invertible.

(14) If o is cancelable, then o°_A is cancelable.

(15) If o has uniquely decomposable unity, then o°_A has uniquely decomposable unity.

(16) If o absorbs o', then o°_A absorbs o'°_A.

(17) Let D_1, D_2, D, E_1, E_2, E be non empty sets, o_1 be a binary function from D_1, D_2 into D, and o_2 be a binary function from E_1, E_2 into E. If $o_1 \leq o_2$, then ${o_1}^\circ_A \leq {o_2}^\circ_A$.

Let G be a non empty groupoid and let A be a set. The functor G^A yields a groupoid and is defined by:

(Def. 3) $G^{A}=\begin{cases}\langle(\text{the carrier of } G)^{A},(\text{the multiplication of } G)^{\circ}_{A},A\longmapsto \mathbf{1}_{\text{the multiplication of } G}\rangle, & \text{if } G \text{is unital},\\ \langle(\text{the carrier of } G)^{A},(\text{the multiplication of } G)^{\circ}_{A}\rangle, & \text{otherwise.}\end{cases}$

Let G be a non empty groupoid and let A be a set. One can check that G^A is non empty.
In the sequel G is a non empty groupoid.
We now state two propositions:

(18)(i) The carrier of $G^X =$ (the carrier of $G)^X$, and
(ii) the multiplication of $G^X =$ (the multiplication of $G)^{\circ}_X$.

(19) x is an element of G^X iff x is a function from X into the carrier of G.

Let G be a non empty groupoid and let A be a set. Observe that G^A is constituted functions.
We now state two propositions:

(20) For every element f of G^X holds $\operatorname{dom} f = X$ and $\operatorname{rng} f \subseteq$ the carrier of G.

(21) For all elements f, g of G^X such that for every x such that $x \in X$ holds $f(x) = g(x)$ holds $f = g$.

Let G be a non empty groupoid, let A be a non empty set, and let f be an element of G^A. Observe that $\operatorname{rng} f$ is non empty. Let a be an element of A. Then $f(a)$ is an element of G.
We now state the proposition

(22) For all elements f_1, f_2 of G^D and for every element a of D holds $(f_1 \cdot f_2)(a) = f_1(a) \cdot f_2(a)$.

Let G be a unital non empty groupoid and let A be a set. Then G^A is a well unital constituted functions strict non empty multiplicative loop structure.
We now state four propositions:

(23) For every unital non empty groupoid G holds the unity of $G^X = X \longmapsto \mathbf{1}_{\text{the multiplication of } G}$.

(24) Let G be a non empty groupoid and A be a set. Then
(i) if G is commutative, then G^A is commutative,
(ii) if G is associative, then G^A is associative,
(iii) if G is idempotent, then G^A is idempotent,
(iv) if G is invertible, then G^A is invertible,
(v) if G is cancelable, then G^A is cancelable, and
(vi) if G has uniquely decomposable unity, then G^A has uniquely decomposable unity.

(25) For every non empty subsystem H of G holds H^X is a subsystem of G^X.

(26) Let G be a unital non empty groupoid and H be a non empty subsystem of G. Suppose $\mathbf{1}_{\text{the multiplication of } G} \in$ the carrier of H. Then H^X is a monoidal subsystem of G^X.

Let G be a unital associative commutative cancelable non empty groupoid with uniquely decomposable unity and let A be a set. Then G^A is a commutative cancelable constituted functions strict monoid with uniquely decomposable unity.

3. MONOID OF MULTISETS OVER A SET

Let A be a set. The functor $A^{\otimes}_{\omega}$ yields a commutative cancelable constituted functions strict monoid with uniquely decomposable unity and is defined as follows:

(Def. 4) $A^{\otimes}_{\omega} = \langle \mathbb{N}, +, 0\rangle^A$.

We now state the proposition

(27) The carrier of $X^{\otimes}_{\omega} = \mathbb{N}^X$ and the multiplication of $X^{\otimes}_{\omega} = (+_{\mathbb{N}})^{\circ}_X$ and the unity of $X^{\otimes}_{\omega} = X \longmapsto 0$.

Let A be a set. A multiset over A is an element of $A^{\otimes}_{\omega}$.

Next we state two propositions:

(28) x is a multiset over X iff x is a function from X into $\mathbb{N}$.

(29) For every multiset m over X holds $\operatorname{dom} m = X$ and $\operatorname{rng} m \subseteq \mathbb{N}$.

Let A be a non empty set and let m be a multiset over A. Then $\operatorname{rng} m$ is a non empty subset of $\mathbb{N}$. Let a be an element of A. Then $m(a)$ is a natural number.

The following two propositions are true:

(30) For all multisets m_1, m_2 over D and for every element a of D holds $(m_1 \otimes m_2)(a) = m_1(a) + m_2(a)$.

(31) $\chi_{Y,X}$ is a multiset over X.

Let us consider Y, X. Then $\chi_{Y,X}$ is a multiset over X.

Let us consider X and let n be a natural number. Then $X \longmapsto n$ is a multiset over X.

Let A be a non empty set and let a be an element of A. The functor χa yielding a multiset over A is defined by:

(Def. 5) $\chi a = \chi_{\{a\},A}$.

The following proposition is true

(32) For every non empty set A and for all elements a, b of A holds $(\chi a)(a) = 1$ and if $b \neq a$, then $(\chi a)(b) = 0$.

For simplicity, we use the following convention: A denotes a non empty set, a denotes an element of A, p denotes a finite sequence of elements of A, and m_1, m_2 denote multisets over A.

We now state the proposition

(33) If for every a holds $m_1(a) = m_2(a)$, then $m_1 = m_2$.

Let A be a set. The functor $A^{\otimes}$ yielding a strict non empty monoidal subsystem of $A^{\otimes}_{\omega}$ is defined by:

(Def. 6) For every multiset f over A holds $f \in$ the carrier of $A^{\otimes}$ iff $f^{-1}(\mathbb{N} \setminus \{0\})$ is finite.

Next we state three propositions:

(34) χa is an element of $A^{\otimes}$.

(35) $\operatorname{dom}(\{x\} \restriction (p \frown \langle x \rangle)) = \operatorname{dom}(\{x\} \restriction p) \cup \{\operatorname{len} p + 1\}$.

(36) If $x \neq y$, then $\operatorname{dom}(\{x\} \restriction (p \frown \langle y \rangle)) = \operatorname{dom}(\{x\} \restriction p)$.

Let A be a set and let F be a finite binary relation. Note that $A \restriction F$ is finite.

Let A be a non empty set and let p be a finite sequence of elements of A. The functor $|p|$ yields a multiset over A and is defined by:

(Def. 7) For every element a of A holds $|p|(a) = \operatorname{card} \operatorname{dom}(\{a\} \restriction p)$.

The following three propositions are true:

(37) $|\varepsilon_A|(a) = 0$.

(38) $|\varepsilon_A| = A \longmapsto 0$.

(39) $|\langle a \rangle| = \chi a$.

In the sequel p, q denote finite sequences of elements of A.

One can prove the following propositions:

(40) $|p \frown \langle a \rangle| = |p| \otimes \chi a$.

(41) $|p \frown q| = |p| \otimes |q|$.

(42) $|n \dot{\longmapsto} a|(a) = n$ and for every element b of A such that $b \neq a$ holds $|n \dot{\longmapsto} a|(b) = 0$.

(43) $|p|$ is an element of $A^{\otimes}$.

(44) If x is an element of $A^{\otimes}$, then there exists p such that $x = |p|$.

4. Monoid of Subsets of a Semigroup

In the sequel a denotes an element of D.

Let D_1, D_2, D be non empty sets and let f be a binary function from D_1, D_2 into D. The functor $^\circ f$ yields a binary function from 2^{D_1}, 2^{D_2} into 2^D and is defined as follows:

(Def. 8) For every element x of $[: 2^{D_1}, 2^{D_2} :]$ holds $(^\circ f)(x) = f^\circ [: x_{\mathbf{1}}, x_{\mathbf{2}} :]$.

One can prove the following propositions:

(45) Let D_1, D_2, D be non empty sets, f be a binary function from D_1, D_2 into D, X_1 be a subset of D_1, and X_2 be a subset of D_2. Then $(^\circ f)(X_1, X_2) = f^\circ [: X_1, X_2 :]$.

(46) Let D_1, D_2, D be non empty sets, f be a binary function from D_1, D_2 into D, X_1 be a subset of D_1, X_2 be a subset of D_2, and x_1, x_2 be sets. If $x_1 \in X_1$ and $x_2 \in X_2$, then $f(x_1, x_2) \in (^\circ f)(X_1, X_2)$.

(47) Let D_1, D_2, D be non empty sets, f be a binary function from D_1, D_2 into D, X_1 be a subset of D_1, and X_2 be a subset of D_2. Then $(^\circ f)(X_1, X_2) = \{f(a, b); a$ ranges over elements of D_1, b ranges over elements of D_2: $a \in X_1 \wedge b \in X_2\}$.

(48) If o is commutative, then $o^\circ [: X, Y :] = o^\circ [: Y, X :]$.

(49) If o is associative, then $o^\circ [: o^\circ [: X, Y :], Z :] = o^\circ [: X, o^\circ [: Y, Z :] :]$.

(50) If o is commutative, then $^\circ o$ is commutative.

(51) If o is associative, then $^\circ o$ is associative.

(52) If a is a unity w.r.t. o, then $o^\circ [: \{a\}, X :] = D \cap X$ and $o^\circ [: X, \{a\} :] = D \cap X$.

(53) If a is a unity w.r.t. o, then $\{a\}$ is a unity w.r.t. $^\circ o$ and $^\circ o$ has a unity and $\mathbf{1}_{^\circ o} = \{a\}$.

(54) If o has a unity, then $^\circ o$ has a unity and $\{\mathbf{1}_o\}$ is a unity w.r.t. $^\circ o$ and $\mathbf{1}_{^\circ o} = \{\mathbf{1}_o\}$.

(55) If o has uniquely decomposable unity, then $^\circ o$ has uniquely decomposable unity.

Let G be a non empty groupoid. The functor 2^G yields a groupoid and is defined by:

(Def. 9) $2^G = \begin{cases} \langle 2^{\text{the carrier of } G}, {^\circ}(\text{the multiplication of } G), \{\mathbf{1}_{\text{the multiplication of } G}\}\rangle, & \text{if } G \text{is unital,} \\ \langle 2^{\text{the carrier of } G}, {^\circ}(\text{the multiplication of } G)\rangle, & \text{otherwise.} \end{cases}$

Let G be a non empty groupoid. One can check that 2^G is non empty.

Let G be a unital non empty groupoid. Then 2^G is a well unital strict non empty multiplicative loop structure.

We now state three propositions:

(56) The carrier of $2^G = 2^{\text{the carrier of } G}$ and the multiplication of $2^G = {^\circ}(\text{the multiplication of } G)$.

(57) For every unital non empty groupoid G holds the unity of $2^G = \{\mathbf{1}_{\text{the multiplication of } G}\}$.

(58) Let G be a non empty groupoid. Then

(i) if G is commutative, then 2^G is commutative,

(ii) if G is associative, then 2^G is associative, and

(iii) if G has uniquely decomposable unity, then 2^G has uniquely decomposable unity.

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