JOURNAL OF FORMALIZED MATHEMATICS
Volume 4, Released 1992, Published 2003
Inst. of Computer Science, Univ. of Białystok

# Opposite Rings, Modules and Their Morphisms

Michał Muzalewski
Warsaw University
Białystok

**Summary.** Let $\mathbb{K} = \langle S; K, 0, 1, +, \cdot \rangle$ be a ring. The structure ${}^{\rm op}\mathbb{K} = \langle S; K, 0, 1, +, \bullet \rangle$ is called anti-ring, if $\alpha \bullet \beta = \beta \cdot \alpha$ for elements $\alpha$, $\beta$ of $K$ [8, pages 5–7]. It is easily seen that ${}^{\rm op}\mathbb{K}$ is also a ring. If $V$ is a left module over $\mathbb{K}$, then $V$ is a right module over ${}^{\rm op}\mathbb{K}$. If $W$ is a right module over $\mathbb{K}$, then $W$ is a left module over ${}^{\rm op}\mathbb{K}$. Let $K, L$ be rings. A morphism $J : K \longrightarrow L$ is called anti-homomorphism, if $J(\alpha \cdot \beta) = J(\beta) \cdot J(\alpha)$ for elements $\alpha$, $\beta$ of $K$. If $J : K \longrightarrow L$ is a homomorphism, then $J : K \longrightarrow {}^{\rm op}L$ is an anti-homomorphism. Let $K, L$ be rings, $V, W$ left modules over $K, L$ respectively and $J : K \longrightarrow L$ an anti-monomorphism. A map $f : V \longrightarrow W$ is called $J$ - semilinear, if $f(x+y) = f(x) + f(y)$ and $f(\alpha \cdot x) = J(\alpha) \cdot f(x)$ for vectors $x, y$ of $V$ and a scalar $\alpha$ of $K$.

MML Identifier: MOD_4.

WWW: http://mizar.org/JFM/Vol4/mod_4.html

The articles [4], [12], [13], [2], [3], [1], [11], [6], [7], [9], [5], and [10] provide the notation and terminology for this paper.

## 1. Opposite functions

In this paper $A$, $B$, $C$ are non empty sets and $f$ is a function from $[:A, B:]$ into $C$.

Let us consider $A$, $B$, $C$, $f$. Then $\curvearrowleft f$ is a function from $[:B, A:]$ into $C$.

One can prove the following proposition

(1) For every element $x$ of $A$ and for every element $y$ of $B$ holds $f(x, y) = (\curvearrowleft f)(y, x)$.

## 2. Opposite rings

In the sequel $K$ is a non empty double loop structure.

Let us consider $K$. The functor ${}^{\rm op}K$ yields a strict double loop structure and is defined as follows:

(Def. 1) ${}^{\rm op}K = \langle$the carrier of $K$, the addition of $K$, $\curvearrowleft$(the multiplication of $K$),the unity of $K$, the zero of $K\rangle$.

Let us consider $K$. One can check that ${}^{\rm op}K$ is non empty.

Let $K$ be an add-associative right complementable right zeroed non empty double loop structure. Observe that ${}^{\rm op}K$ is add-associative, right zeroed, and right complementable.

One can prove the following propositions:

(2)(i) The loop structure of ${}^{\rm op}K =$ the loop structure of $K$,

(ii) if $K$ is add-associative, right zeroed, and right complementable, then $\mathrm{comp}\,{}^{\rm op}K = \mathrm{comp}\,K$, and

(iii) for every set $x$ holds $x$ is a scalar of ${}^{\rm op}K$ iff $x$ is a scalar of $K$.

(3) $^{\rm op}(^{\rm op}K)$ = the double loop structure of $K$.

(4) Let $K$ be an add-associative right zeroed right complementable non empty double loop structure. Then

(i) $0_K = 0_{^{\rm op}K}$,

(ii) $\mathbf{1}_K = \mathbf{1}_{^{\rm op}K}$, and

(iii) for all scalars $x$, $y$, $z$, $u$ of $K$ and for all scalars $a$, $b$, $c$, $d$ of $^{\rm op}K$ such that $x = a$ and $y = b$ and $z = c$ and $u = d$ holds $x + y = a + b$ and $x \cdot y = b \cdot a$ and $-x = -a$ and $x + y + z = a + b + c$ and $x + (y + z) = a + (b + c)$ and $(x \cdot y) \cdot z = c \cdot (b \cdot a)$ and $x \cdot (y \cdot z) = (c \cdot b) \cdot a$ and $x \cdot (y + z) = (b + c) \cdot a$ and $(y + z) \cdot x = a \cdot (b + c)$ and $x \cdot y + z \cdot u = b \cdot a + d \cdot c$.

(5) For every ring $K$ holds $^{\rm op}K$ is a strict ring.

Let $K$ be a ring. One can check that $^{\rm op}K$ is Abelian, add-associative, right zeroed, right complementable, well unital, and distributive.

One can prove the following proposition

(6) For every ring $K$ holds $^{\rm op}K$ is a ring.

Let $K$ be a ring. One can verify that $^{\rm op}K$ is associative.

We now state the proposition

(7) For every skew field $K$ holds $^{\rm op}K$ is a skew field.

Let $K$ be a skew field. One can check that $^{\rm op}K$ is non degenerated, field-like, associative, Abelian, add-associative, right zeroed, right complementable, well unital, and distributive.

One can prove the following proposition

(8) For every field $K$ holds $^{\rm op}K$ is a strict field.

Let $K$ be a field. Observe that $^{\rm op}K$ is strict and field-like.

## 3. Opposite modules

In the sequel $V$ is a non empty vector space structure over $K$.

Let us consider $K$, $V$. The functor $^{\rm op}V$ yields a strict right module structure over $^{\rm op}K$ and is defined by the condition (Def. 2).

(Def. 2) Let $o$ be a function from [: the carrier of $V$, the carrier of $^{\rm op}K$ :] into the carrier of $V$. Suppose $o = \curvearrowleft$(the left multiplication of $V$). Then $^{\rm op}V = \langle$the carrier of $V$, the addition of $V$, the zero of $V$, $o\rangle$.

Let us consider $K$, $V$. Observe that $^{\rm op}V$ is non empty.

We now state the proposition

(9)(i) The loop structure of $^{\rm op}V$ = the loop structure of $V$, and

(ii) for every set $x$ holds $x$ is a vector of $V$ iff $x$ is a vector of $^{\rm op}V$.

Let us consider $K$, $V$ and let $o$ be a function from [: the carrier of $K$, the carrier of $V$ :] into the carrier of $V$. The functor $^{\rm op}o$ yielding a function from [: the carrier of $^{\rm op}V$, the carrier of $^{\rm op}K$ :] into the carrier of $^{\rm op}V$ is defined by:

(Def. 3) $^{\rm op}o = \curvearrowleft o$.

One can prove the following proposition

(10) The right multiplication of $^{\rm op}V$ = $^{\rm op}$(the left multiplication of $V$).

In the sequel $W$ is a non empty right module structure over $K$.

Let us consider $K$, $W$. The functor ${}^{\rm op}W$ yields a strict vector space structure over ${}^{\rm op}K$ and is defined by the condition (Def. 4).

(Def. 4) Let $o$ be a function from [: the carrier of ${}^{\rm op}K$, the carrier of $W$ :] into the carrier of $W$. Suppose $o = \curvearrowleft$(the right multiplication of $W$). Then ${}^{\rm op}W = \langle$the carrier of $W$, the addition of $W$, the zero of $W$, $o\rangle$.

Let us consider $K$, $W$. One can check that ${}^{\rm op}W$ is non empty.

The following proposition is true

(12)[1] (i) The loop structure of ${}^{\rm op}W =$ the loop structure of $W$, and

(ii) for every set $x$ holds $x$ is a vector of $W$ iff $x$ is a vector of ${}^{\rm op}W$.

Let us consider $K$, $W$ and let $o$ be a function from [: the carrier of $W$, the carrier of $K$ :] into the carrier of $W$. The functor ${}^{\rm op}o$ yields a function from [: the carrier of ${}^{\rm op}K$, the carrier of ${}^{\rm op}W$ :] into the carrier of ${}^{\rm op}W$ and is defined as follows:

(Def. 5) ${}^{\rm op}o = \curvearrowleft o$.

We now state a number of propositions:

(13) The left multiplication of ${}^{\rm op}W = {}^{\rm op}$(the right multiplication of $W$).

(15)[2] For every function $o$ from [: the carrier of $K$, the carrier of $V$ :] into the carrier of $V$ holds ${}^{\rm op}({}^{\rm op}o) = o$.

(16) Let $o$ be a function from [: the carrier of $K$, the carrier of $V$ :] into the carrier of $V$, $x$ be a scalar of $K$, $y$ be a scalar of ${}^{\rm op}K$, $v$ be a vector of $V$, and $w$ be a vector of ${}^{\rm op}V$. If $x = y$ and $v = w$, then $({}^{\rm op}o)(w, y) = o(x, v)$.

(17) Let $K$, $L$ be rings, $V$ be a non empty vector space structure over $K$, $W$ be a non empty right module structure over $L$, $x$ be a scalar of $K$, $y$ be a scalar of $L$, $v$ be a vector of $V$, and $w$ be a vector of $W$. If $L = {}^{\rm op}K$ and $W = {}^{\rm op}V$ and $x = y$ and $v = w$, then $w \cdot y = x \cdot v$.

(18) Let $K$, $L$ be rings, $V$ be a non empty vector space structure over $K$, $W$ be a non empty right module structure over $L$, $v_1$, $v_2$ be vectors of $V$, and $w_1$, $w_2$ be vectors of $W$. If $L = {}^{\rm op}K$ and $W = {}^{\rm op}V$ and $v_1 = w_1$ and $v_2 = w_2$, then $w_1 + w_2 = v_1 + v_2$.

(19) For every function $o$ from [: the carrier of $W$, the carrier of $K$ :] into the carrier of $W$ holds ${}^{\rm op}({}^{\rm op}o) = o$.

(20) Let $o$ be a function from [: the carrier of $W$, the carrier of $K$ :] into the carrier of $W$, $x$ be a scalar of $K$, $y$ be a scalar of ${}^{\rm op}K$, $v$ be a vector of $W$, and $w$ be a vector of ${}^{\rm op}W$. If $x = y$ and $v = w$, then $({}^{\rm op}o)(y, w) = o(v, x)$.

(21) Let $K$, $L$ be rings, $V$ be a non empty vector space structure over $K$, $W$ be a non empty right module structure over $L$, $x$ be a scalar of $K$, $y$ be a scalar of $L$, $v$ be a vector of $V$, and $w$ be a vector of $W$. If $K = {}^{\rm op}L$ and $V = {}^{\rm op}W$ and $x = y$ and $v = w$, then $w \cdot y = x \cdot v$.

(22) Let $K$, $L$ be rings, $V$ be a non empty vector space structure over $K$, $W$ be a non empty right module structure over $L$, $v_1$, $v_2$ be vectors of $V$, and $w_1$, $w_2$ be vectors of $W$. If $K = {}^{\rm op}L$ and $V = {}^{\rm op}W$ and $v_1 = w_1$ and $v_2 = w_2$, then $w_1 + w_2 = v_1 + v_2$.

(23) Let $K$ be a strict non empty double loop structure and $V$ be a non empty vector space structure over $K$. Then ${}^{\rm op}({}^{\rm op}V) =$ the vector space structure of $V$.

(24) Let $K$ be a strict non empty double loop structure and $W$ be a non empty right module structure over $K$. Then ${}^{\rm op}({}^{\rm op}W) =$ the right module structure of $W$.

[1] The proposition (11) has been removed.

[2] The proposition (14) has been removed.

(25) For every ring $K$ and for every left module $V$ over $K$ holds ${}^{\rm op}V$ is a strict right module over ${}^{\rm op}K$.

Let $K$ be a ring and let $V$ be a left module over $K$. Observe that ${}^{\rm op}V$ is Abelian, add-associative, right zeroed, right complementable, and right module-like.

The following proposition is true

(26) For every ring $K$ and for every right module $W$ over $K$ holds ${}^{\rm op}W$ is a strict left module over ${}^{\rm op}K$.

Let $K$ be a ring and let $W$ be a right module over $K$. Observe that ${}^{\rm op}W$ is Abelian, add-associative, right zeroed, right complementable, and vector space-like.

## 4. Morphisms of rings

Let $K$, $L$ be non empty double loop structures and let $I_1$ be a map from $K$ into $L$. We say that $I_1$ is antilinear if and only if:

(Def. 6) For all scalars $x$, $y$ of $K$ holds $I_1(x+y) = I_1(x)+I_1(y)$ and for all scalars $x$, $y$ of $K$ holds $I_1(x\cdot y) = I_1(y)\cdot I_1(x)$ and $I_1(\mathbf{1}_K) = \mathbf{1}_L$.

Let $K$, $L$ be non empty double loop structures and let $I_1$ be a map from $K$ into $L$. We say that $I_1$ is monomorphism if and only if:

(Def. 7) $I_1$ is linear and one-to-one.

We say that $I_1$ is antimonomorphism if and only if:

(Def. 8) $I_1$ is antilinear and one-to-one.

Let $K$, $L$ be non empty double loop structures and let $I_1$ be a map from $K$ into $L$. We say that $I_1$ is epimorphism if and only if:

(Def. 9) $I_1$ is linear and $\mathrm{rng}\, I_1 =$ the carrier of $L$.

We say that $I_1$ is antiepimorphism if and only if:

(Def. 10) $I_1$ is antilinear and $\mathrm{rng}\, I_1 =$ the carrier of $L$.

Let $K$, $L$ be non empty double loop structures and let $I_1$ be a map from $K$ into $L$. We say that $I_1$ is isomorphism if and only if:

(Def. 11) $I_1$ is monomorphism and $\mathrm{rng}\, I_1 =$ the carrier of $L$.

We say that $I_1$ is antiisomorphism if and only if:

(Def. 12) $I_1$ is antimonomorphism and $\mathrm{rng}\, I_1 =$ the carrier of $L$.

In the sequel $J$ is a map from $K$ into $K$.

Let $K$ be a non empty double loop structure and let $I_1$ be a map from $K$ into $K$. We say that $I_1$ is endomorphism if and only if:

(Def. 13) $I_1$ is linear.

We say that $I_1$ is antiendomorphism if and only if:

(Def. 14) $I_1$ is antilinear.

We say that $I_1$ is automorphism if and only if:

(Def. 15) $I_1$ is isomorphism.

We say that $I_1$ is antiautomorphism if and only if:

(Def. 16) $I_1$ is antiisomorphism.

Next we state three propositions:

(27) $J$ is automorphism if and only if the following conditions are satisfied:
(i) for all scalars $x$, $y$ of $K$ holds $J(x+y) = J(x) + J(y)$,
(ii) for all scalars $x$, $y$ of $K$ holds $J(x \cdot y) = J(x) \cdot J(y)$,
(iii) $J(\mathbf{1}_K) = \mathbf{1}_K$,
(iv) $J$ is one-to-one, and
(v) $\mathrm{rng}\, J =$ the carrier of $K$.

(28) $J$ is antiautomorphism if and only if the following conditions are satisfied:
(i) for all scalars $x$, $y$ of $K$ holds $J(x+y) = J(x) + J(y)$,
(ii) for all scalars $x$, $y$ of $K$ holds $J(x \cdot y) = J(y) \cdot J(x)$,
(iii) $J(\mathbf{1}_K) = \mathbf{1}_K$,
(iv) $J$ is one-to-one, and
(v) $\mathrm{rng}\, J =$ the carrier of $K$.

(29) $\mathrm{id}_K$ is automorphism.

We adopt the following rules: $K$, $L$ denote rings, $J$ denotes a map from $K$ into $L$, and $x$, $y$ denote scalars of $K$.

Next we state four propositions:

(30) If $J$ is linear, then $J(0_K) = 0_L$ and $J(-x) = -J(x)$ and $J(x-y) = J(x) - J(y)$.

(31) If $J$ is antilinear, then $J(0_K) = 0_L$ and $J(-x) = -J(x)$ and $J(x-y) = J(x) - J(y)$.

(32) For every ring $K$ holds $\mathrm{id}_K$ is antiautomorphism iff $K$ is a commutative ring.

(33) For every skew field $K$ holds $\mathrm{id}_K$ is antiautomorphism iff $K$ is a field.

## 5. Opposite morphisms to morphisms of rings

Let $K$, $L$ be non empty double loop structures and let $J$ be a map from $K$ into $L$. The functor ${}^{\mathrm{op}}J$ yields a map from $K$ into ${}^{\mathrm{op}}L$ and is defined as follows:

(Def. 17) ${}^{\mathrm{op}}J = J$.

In the sequel $K$, $L$ are add-associative right zeroed right complementable non empty double loop structures and $J$ is a map from $K$ into $L$.

One can prove the following propositions:

(34) ${}^{\mathrm{op}}({}^{\mathrm{op}}J) = J$.

(35) Let $K$, $L$ be add-associative right zeroed right complementable non empty double loop structures and $J$ be a map from $K$ into $L$. Then $J$ is linear if and only if ${}^{\mathrm{op}}J$ is antilinear.

(36) $J$ is antilinear iff ${}^{\mathrm{op}}J$ is linear.

(37) $J$ is monomorphism iff ${}^{\mathrm{op}}J$ is antimonomorphism.

(38) $J$ is antimonomorphism iff ${}^{\mathrm{op}}J$ is monomorphism.

(39) $J$ is epimorphism iff ${}^{\mathrm{op}}J$ is antiepimorphism.

(40) $J$ is antiepimorphism iff ${}^{\mathrm{op}}J$ is epimorphism.

(41) $J$ is isomorphism iff ${}^{\mathrm{op}}J$ is antiisomorphism.

(42) $J$ is antiisomorphism iff ${}^{\rm op}J$ is isomorphism.

In the sequel $J$ denotes a map from $K$ into $K$.
Next we state four propositions:

(43) $J$ is endomorphism iff ${}^{\rm op}J$ is antilinear.

(44) $J$ is antiendomorphism iff ${}^{\rm op}J$ is linear.

(45) $J$ is automorphism iff ${}^{\rm op}J$ is antiisomorphism.

(46) $J$ is antiautomorphism iff ${}^{\rm op}J$ is isomorphism.

## 6. MORPHISMS OF GROUPS

In the sequel $G$, $H$ are groups.

Let us consider $G$, $H$. A map from $G$ into $H$ is said to be a homomorphism from $G$ to $H$ if:

(Def. 18) For all elements $x$, $y$ of $G$ holds $\mathrm{it}(x+y) = \mathrm{it}(x) + \mathrm{it}(y)$.

Let us consider $G$, $H$. Then $\mathrm{ZeroMap}(G,H)$ is a homomorphism from $G$ to $H$.
In the sequel $f$ denotes a homomorphism from $G$ to $H$.
Let us consider $G$, $H$ and let $I_1$ be a homomorphism from $G$ to $H$. We say that $I_1$ is monomorphism if and only if:

(Def. 19) $I_1$ is one-to-one.

Let us consider $G$, $H$ and let $I_1$ be a homomorphism from $G$ to $H$. We say that $I_1$ is epimorphism if and only if:

(Def. 20) $\mathrm{rng}\, I_1 =$ the carrier of $H$.

Let us consider $G$, $H$ and let $I_1$ be a homomorphism from $G$ to $H$. We say that $I_1$ is isomorphism if and only if:

(Def. 21) $I_1$ is one-to-one and $\mathrm{rng}\, I_1 =$ the carrier of $H$.

Let us consider $G$. An endomorphism of $G$ is a homomorphism from $G$ to $G$.
Let us consider $G$. Observe that there exists an endomorphism of $G$ which is isomorphism.
Let us consider $G$. An automorphism of $G$ is an isomorphism endomorphism of $G$.
Let us consider $G$. Then $\mathrm{id}_G$ is an automorphism of $G$.
In the sequel $x$, $y$ denote elements of $G$.
The following proposition is true

(48)[3] $f(0_G) = 0_H$ and $f(-x) = -f(x)$ and $f(x-y) = f(x) - f(y)$.

We adopt the following convention: $G$, $H$ denote Abelian groups, $f$ denotes a homomorphism from $G$ to $H$, and $x$, $y$ denote elements of $G$.
The following proposition is true

(49) $f(x-y) = f(x) - f(y)$.

[3] The proposition (47) has been removed.

## 7. SEMILINEAR MORPHISMS

For simplicity, we adopt the following convention: $K$, $L$ are rings, $J$ is a map from $K$ into $L$, $V$ is a left module over $K$, and $W$ is a left module over $L$.

Let us consider $K$, $L$, $J$, $V$, $W$. A map from $V$ into $W$ is said to be a homomorphism from $V$ to $W$ by $J$ if it satisfies the conditions (Def. 23).

(Def. 23)[4] (i) For all vectors $x$, $y$ of $V$ holds $\mathrm{it}(x+y) = \mathrm{it}(x) + \mathrm{it}(y)$, and

(ii) for every scalar $a$ of $K$ and for every vector $x$ of $V$ holds $\mathrm{it}(a \cdot x) = J(a) \cdot \mathrm{it}(x)$.

Next we state the proposition

(50) $\mathrm{ZeroMap}(V, W)$ is a homomorphism from $V$ to $W$ by $J$.

In the sequel $f$ is a homomorphism from $V$ to $W$ by $J$.

Let us consider $K$, $L$, $J$, $V$, $W$, $f$. We say that $f$ is a monomorphism wrp $J$ if and only if:

(Def. 24) $f$ is one-to-one.

We say that $f$ is an epimorphism wrp $J$ if and only if:

(Def. 25) $\mathrm{rng}\, f =$ the carrier of $W$.

We say that $f$ is an isomorphism wrp $J$ if and only if:

(Def. 26) $f$ is one-to-one and $\mathrm{rng}\, f =$ the carrier of $W$.

In the sequel $J$ is a map from $K$ into $K$ and $f$ is a homomorphism from $V$ to $V$ by $J$.

Let us consider $K$, $J$, $V$. An endomorphism of $J$ and $V$ is a homomorphism from $V$ to $V$ by $J$.

Let us consider $K$, $J$, $V$, $f$. We say that $f$ is an automorphism wrp $J$ if and only if:

(Def. 27) $f$ is one-to-one and $\mathrm{rng}\, f =$ the carrier of $V$.

In the sequel $W$ is a left module over $K$.

Let us consider $K$, $V$, $W$. A homomorphism from $V$ to $W$ is a homomorphism from $V$ to $W$ by $\mathrm{id}_K$.

One can prove the following proposition

(51) Let $f$ be a map from $V$ into $W$. Then $f$ is a homomorphism from $V$ to $W$ if and only if the following conditions are satisfied:

(i) for all vectors $x$, $y$ of $V$ holds $f(x+y) = f(x) + f(y)$, and

(ii) for every scalar $a$ of $K$ and for every vector $x$ of $V$ holds $f(a \cdot x) = a \cdot f(x)$.

Let us consider $K$, $V$, $W$ and let $I_1$ be a homomorphism from $V$ to $W$. We say that $I_1$ is monomorphism if and only if:

(Def. 28) $I_1$ is one-to-one.

We say that $I_1$ is epimorphism if and only if:

(Def. 29) $\mathrm{rng}\, I_1 =$ the carrier of $W$.

We say that $I_1$ is isomorphism if and only if:

(Def. 30) $I_1$ is one-to-one and $\mathrm{rng}\, I_1 =$ the carrier of $W$.

Let us consider $K$, $V$. An endomorphism of $V$ is a homomorphism from $V$ to $V$.

Let us consider $K$, $V$ and let $I_1$ be an endomorphism of $V$. We say that $I_1$ is automorphism if and only if:

(Def. 31) $I_1$ is one-to-one and $\mathrm{rng}\, I_1 =$ the carrier of $V$.

[4] The definition (Def. 22) has been removed.

## REFERENCES

[1] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[4] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[5] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_4.html.

[6] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[7] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_2.html.

[8] Michał Muzalewski. *Foundations of Metric-Affine Geometry*. Dział Wydawnictw Filii UW w Białymstoku, Filia UW w Białymstoku, 1990.

[9] Michał Muzalewski. Categories of groups. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/grcat_1.html.

[10] Michał Muzalewski. Category of rings. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/ringcat1.html.

[11] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[12] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[13] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

*Received June 22, 1992*

*Published January 2, 2004*