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Free Modules

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Summary. We define free modules and prove that every left module over Skew-Field is free.

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The articles [10], [5], [17], [6], [2], [18], [3], [4], [11], [12], [1], [13], [7], [8], [16], [15], [14], and [9] provide the notation and terminology for this paper.

One can prove the following proposition

(2)[1] Let R be a non degenerated add-associative right zeroed right complementable non empty double loop structure. Then $0_R \neq -\mathbf{1}_R$.

For simplicity, we adopt the following rules: x denotes a set, R denotes a ring, V denotes a left module over R, L denotes a linear combination of V, a denotes a scalar of R, F denotes a finite sequence of elements of the carrier of V, and C denotes a finite subset of V.

One can prove the following two propositions:

(6)[2] If the support of $L \subseteq C$, then there exists F such that F is one-to-one and $\operatorname{rng} F = C$ and $\sum L = \sum (L\, F)$.

(7) $\sum (a \cdot L) = a \cdot \sum L$.

In the sequel A, B are subsets of V and l is a linear combination of A.

Let us consider R, V, A. The functor $\mathrm{Lin}(A)$ yielding a strict subspace of V is defined by:

(Def. 1) The carrier of $\mathrm{Lin}(A) = \{\sum l\}$.

One can prove the following propositions:

(11)[3] $x \in \mathrm{Lin}(A)$ iff there exists l such that $x = \sum l$.

(12) If $x \in A$, then $x \in \mathrm{Lin}(A)$.

(13) $\mathrm{Lin}(\emptyset_{\text{the carrier of } V}) = \mathbf{0}_V$.

(14) If $\mathrm{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{0_V\}$.

(15) For every strict subspace W of V such that $0_R \neq \mathbf{1}_R$ and $A =$ the carrier of W holds $\mathrm{Lin}(A) = W$.

[1] The proposition (1) has been removed.
[2] The propositions (3)–(5) have been removed.
[3] The propositions (8)–(10) have been removed.

(16) Let V be a strict left module over R and A be a subset of V. If $0_R \neq \mathbf{1}_R$ and $A =$ the carrier of V, then $\mathrm{Lin}(A) = V$.

(17) If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(18) If $\mathrm{Lin}(A) = V$ and $A \subseteq B$, then $\mathrm{Lin}(B) = V$.

(19) $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(20) $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

Let us consider R, V and let I_1 be a subset of V. We say that I_1 is base if and only if:

(Def. 2) I_1 is linearly independent and $\mathrm{Lin}(I_1) =$ the vector space structure of V.

Let us consider R and let I_1 be a left module over R. We say that I_1 is free if and only if:

(Def. 3) There exists a subset of I_1 which is base.

The following proposition is true

(21) $\mathbf{0}_V$ is free.

Let us consider R. Observe that there exists a left module over R which is strict and free.

For simplicity, we adopt the following rules: R is a skew field, a, b are scalars of R, V is a left module over R, and v, v_1, v_2 are vectors of V.

The following propositions are true:

(23)[4] $\{v\}$ is linearly independent iff $v \neq 0_V$.

(24) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent iff $v_2 \neq 0_V$ and for every a holds $v_1 \neq a \cdot v_2$.

(25) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent iff for all a, b such that $a \cdot v_1 + b \cdot v_2 = 0_V$ holds $a = 0_R$ and $b = 0_R$.

(26) Let V be a left module over R and A be a subset of V. Suppose A is linearly independent. Then there exists a subset B of V such that $A \subseteq B$ and B is base.

(27) Let V be a left module over R and A be a subset of V. If $\mathrm{Lin}(A) = V$, then there exists a subset B of V such that $B \subseteq A$ and B is base.

(28) Every left module over R is free.

Let us consider R and let V be a left module over R. A subset of V is called a basis of V if:

(Def. 5)[5] It is base.

One can prove the following two propositions:

(29) Let V be a left module over R and A be a subset of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

(30) For every left module V over R and for every subset A of V such that $\mathrm{Lin}(A) = V$ there exists a basis I of V such that $I \subseteq A$.

[4] The proposition (22) has been removed.
[5] The definition (Def. 4) has been removed.

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