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Metrics in Cartesian Product[1]

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Summary. A continuation of paper [6]. It deals with the method of creation of the distance in the Cartesian product of metric spaces. The distance of two points belonging to Cartesian product of metric spaces has been defined as sum of distances of appropriate coordinates (or projections) of these points. It is shown that product of metric spaces with such a distance is a metric space.

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The articles [7], [4], [10], [9], [5], [2], [3], [1], [6], and [8] provide the notation and terminology for this paper.

We adopt the following rules: X, Y denote non empty metric spaces, x_1, y_1, z_1 denote elements of X, and x_2, y_2, z_2 denote elements of Y.

The scheme *LambdaMCART* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a 4-ary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

> There exists a function f from $[:[:\mathcal{A},\mathcal{B}:],[:\mathcal{A},\mathcal{B}:]:]$ into $\mathcal{C}$ such that for all elements x_1, y_1 of $\mathcal{A}$ and for all elements x_2, y_2 of $\mathcal{B}$ and for all elements x, y of $[:\mathcal{A},\mathcal{B}:]$ if $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$, then $f(\langle x, y\rangle) = \mathcal{F}(x_1, y_1, x_2, y_2)$

for all values of the parameters.

Let us consider X, Y. The functor $\rho^{X\times Y}$ yielding a function from $[:[:$the carrier of X, the carrier of $Y:], [:$the carrier of X, the carrier of $Y:]:]$ into $\mathbb{R}$ is defined by the condition (Def. 1).

(Def. 1) Let x_1, y_1 be elements of X, x_2, y_2 be elements of Y, and x, y be elements of $[:$the carrier of X, the carrier of $Y:]$. If $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$, then $\rho^{X\times Y}(x, y) = \rho(x_1, y_1) + \rho(x_2, y_2)$.

One can prove the following propositions:

(2)[1] For all elements a, b of $\mathbb{R}$ such that $a + b = 0$ and $0 \leq a$ and $0 \leq b$ holds $a = 0$ and $b = 0$.

(5)[2] Let x, y be elements of $[:$the carrier of X, the carrier of $Y:]$. If $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$, then $\rho^{X\times Y}(x, y) = 0$ iff $x = y$.

(6) For all elements x, y of $[:$the carrier of X, the carrier of $Y:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{X\times Y}(x, y) = \rho^{X\times Y}(y, x)$.

(7) Let x, y, z be elements of $[:$the carrier of X, the carrier of $Y:]$. If $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ and $z = \langle z_1, z_2\rangle$, then $\rho^{X\times Y}(x, z) \leq \rho^{X\times Y}(x, y) + \rho^{X\times Y}(y, z)$.

Let us consider X, Y and let x, y be elements of $[:$the carrier of X, the carrier of $Y:]$. The functor $\rho(x, y)$ yielding a real number is defined as follows:

[1]Supported by RPBP.III-24.B3.
[1] The proposition (1) has been removed.
[2] The propositions (3) and (4) have been removed.

(Def. 2) $\rho(x,y) = \rho^{X \times Y}(x,y)$.

Let A be a non empty set and let r be a function from $[: A, A :]$ into $\mathbb{R}$. Observe that $\langle A, r \rangle$ is non empty.

Let us consider X, Y. The functor $[: X, Y :]$ yields a strict non empty metric space and is defined as follows:

(Def. 3) $[: X, Y :] = \langle [:$ the carrier of X, the carrier of $Y :], \rho^{X \times Y} \rangle$.

One can prove the following proposition

(9)[3] $\langle [:$ the carrier of X, the carrier of $Y :], \rho^{X \times Y} \rangle$ is a metric space.

In the sequel Z is a non empty metric space and x_3, y_3, z_3 are elements of Z.

The scheme *LambdaMCART1* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and a 6-ary functor $\mathcal{F}$ yielding an element of $\mathcal{D}$, and states that:

> There exists a function f from $[: [: \mathcal{A}, \mathcal{B}, \mathcal{C} :], [: \mathcal{A}, \mathcal{B}, \mathcal{C} :] :]$ into $\mathcal{D}$ such that for all elements x_1, y_1 of $\mathcal{A}$ and for all elements x_2, y_2 of $\mathcal{B}$ and for all elements x_3, y_3 of $\mathcal{C}$ and for all elements x, y of $[: \mathcal{A}, \mathcal{B}, \mathcal{C} :]$ if $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$, then $f(\langle x, y \rangle) = \mathcal{F}(x_1, y_1, x_2, y_2, x_3, y_3)$

for all values of the parameters.

Let us consider X, Y, Z. The functor $\rho^{X \times Y \times Z}$ yields a function from $[: [:$ the carrier of X, the carrier of Y, the carrier of $Z :], [:$ the carrier of X, the carrier of Y, the carrier of $Z :] :]$ into $\mathbb{R}$ and is defined by the condition (Def. 4).

(Def. 4) Let x_1, y_1 be elements of X, x_2, y_2 be elements of Y, x_3, y_3 be elements of Z, and x, y be elements of $[:$ the carrier of X, the carrier of Y, the carrier of $Z :]$. If $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$, then $\rho^{X \times Y \times Z}(x,y) = \rho(x_1,y_1) + \rho(x_2,y_2) + \rho(x_3,y_3)$.

We now state three propositions:

(12)[4] Let x, y be elements of $[:$ the carrier of X, the carrier of Y, the carrier of $Z :]$. If $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$, then $\rho^{X \times Y \times Z}(x,y) = 0$ iff $x = y$.

(13) Let x, y be elements of $[:$ the carrier of X, the carrier of Y, the carrier of $Z :]$. If $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$, then $\rho^{X \times Y \times Z}(x,y) = \rho^{X \times Y \times Z}(y,x)$.

(14) Let x, y, z be elements of $[:$ the carrier of X, the carrier of Y, the carrier of $Z :]$. If $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ and $z = \langle z_1, z_2, z_3 \rangle$, then $\rho^{X \times Y \times Z}(x,z) \leq \rho^{X \times Y \times Z}(x,y) + \rho^{X \times Y \times Z}(y,z)$.

Let us consider X, Y, Z. The functor $[: X, Y, Z :]$ yielding a strict non empty metric space is defined as follows:

(Def. 5) $[: X, Y, Z :] = \langle [:$ the carrier of X, the carrier of Y, the carrier of $Z :], \rho^{X \times Y \times Z} \rangle$.

Let us consider X, Y, Z and let x, y be elements of $[:$ the carrier of X, the carrier of Y, the carrier of $Z :]$. The functor $\rho(x,y)$ yields a real number and is defined as follows:

(Def. 6) $\rho(x,y) = \rho^{X \times Y \times Z}(x,y)$.

The following proposition is true

(16)[5] $\langle [:$ the carrier of X, the carrier of Y, the carrier of $Z :], \rho^{X \times Y \times Z} \rangle$ is a metric space.

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[3] The proposition (8) has been removed.
[4] The propositions (10) and (11) have been removed.
[5] The proposition (15) has been removed.

In the sequel W denotes a non empty metric space and x_4, y_4, z_4 denote elements of W.

The scheme *LambdaMCART2* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and a 8-ary functor $\mathcal{F}$ yielding an element of $\mathcal{E}$, and states that:

> There exists a function f from $[:[:\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}:], [:\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}:]:]$ into $\mathcal{E}$ such that for all elements x_1, y_1 of $\mathcal{A}$ and for all elements x_2, y_2 of $\mathcal{B}$ and for all elements x_3, y_3 of $\mathcal{C}$ and for all elements x_4, y_4 of $\mathcal{D}$ and for all elements x, y of $[:\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}:]$ if $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$, then $f(\langle x, y \rangle) = \mathcal{F}(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$

for all values of the parameters.

Let us consider X, Y, Z, W. The functor $\rho^{X \times Y \times Z \times W}$ yields a function from $[:[:$the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W:]$, $[:$the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W:]:]$ into $\mathbb{R}$ and is defined by the condition (Def. 7).

(Def. 7) Let x_1, y_1 be elements of X, x_2, y_2 be elements of Y, x_3, y_3 be elements of Z, x_4, y_4 be elements of W, and x, y be elements of $[:$the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W:]$. If $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$, then $\rho^{X \times Y \times Z \times W}(x, y) = \rho(x_1, y_1) + \rho(x_2, y_2) + (\rho(x_3, y_3) + \rho(x_4, y_4))$.

Next we state three propositions:

(19)[6] Let x, y be elements of $[:$the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W:]$. If $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$, then $\rho^{X \times Y \times Z \times W}(x, y) = 0$ iff $x = y$.

(20) Let x, y be elements of $[:$the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W:]$. If $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$, then $\rho^{X \times Y \times Z \times W}(x, y) = \rho^{X \times Y \times Z \times W}(y, x)$.

(21) Let x, y, z be elements of $[:$the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W:]$. Suppose $x = \langle x_1, x_2, x_3, x_4 \rangle$ and $y = \langle y_1, y_2, y_3, y_4 \rangle$ and $z = \langle z_1, z_2, z_3, z_4 \rangle$. Then $\rho^{X \times Y \times Z \times W}(x, z) \leq \rho^{X \times Y \times Z \times W}(x, y) + \rho^{X \times Y \times Z \times W}(y, z)$.

Let us consider X, Y, Z, W. The functor $[:X, Y, Z, W:]$ yields a strict non empty metric space and is defined as follows:

(Def. 8) $[:X, Y, Z, W:] = \langle [:$the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W:], \rho^{X \times Y \times Z \times W} \rangle$.

Let us consider X, Y, Z, W and let x, y be elements of $[:$the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W:]$. The functor $\rho(x, y)$ yields a real number and is defined as follows:

(Def. 9) $\rho(x, y) = \rho^{X \times Y \times Z \times W}(x, y)$.

We now state the proposition

(23)[7] $\langle [:$the carrier of X, the carrier of Y, the carrier of Z, the carrier of $W:], \rho^{X \times Y \times Z \times W} \rangle$ is a metric space.

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