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Metric Spaces[1]

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Summary. In this paper we define the metric spaces. Two examples of metric spaces are given. We define the discrete metric and the metric on the real axis. Moreover the open ball, the close ball and the sphere in metric spaces are introduced. We also prove some theorems concerning these concepts.

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The articles [9], [5], [11], [1], [10], [6], [3], [4], [2], [8], [12], and [7] provide the notation and terminology for this paper.

We introduce metric structures which are extensions of 1-sorted structure and are systems
⟨ a carrier, a distance ⟩,
where the carrier is a set and the distance is a function from [:the carrier, the carrier:] into $\mathbb{R}$.

One can check that there exists a metric structure which is non empty and strict.

Let A, B be sets, let f be a partial function from $[:A, B:]$ to $\mathbb{R}$, let a be an element of A, and let b be an element of B. Then $f(a, b)$ is a real number.

Let M be a metric structure and let a, b be elements of M. The functor $\rho(a,b)$ yields a real number and is defined by:

(Def. 1) $\rho(a,b) = (\text{the distance of } M)(a, b)$.

The function $\{[\emptyset,\emptyset]\} \mapsto 0$ from $[:\{\emptyset\}, \{\emptyset\}:]$ into $\mathbb{R}$ is defined as follows:

(Def. 2) $\{[\emptyset,\emptyset]\} \mapsto 0 = [:\{\emptyset\}, \{\emptyset\}:] \longmapsto 0$.

Let A be a set and let f be a partial function from $[:A, A:]$ to $\mathbb{R}$. We say that f is Reflexive if and only if:

(Def. 3) For every element a of A holds $f(a, a) = 0$.

We say that f is discernible if and only if:

(Def. 4) For all elements a, b of A such that $f(a, b) = 0$ holds $a = b$.

We say that f is symmetric if and only if:

(Def. 5) For all elements a, b of A holds $f(a, b) = f(b, a)$.

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We say that f is triangle if and only if:

(Def. 6) For all elements a, b, c of A holds $f(a, c) \leq f(a, b) + f(b, c)$.

Let M be a metric structure. We say that M is Reflexive if and only if:

(Def. 7) The distance of M is Reflexive.

We say that M is discernible if and only if:

(Def. 8) The distance of M is discernible.

We say that M is symmetric if and only if:

(Def. 9) The distance of M is symmetric.

We say that M is triangle if and only if:

(Def. 10) The distance of M is triangle.

Let us observe that there exists a metric structure which is strict, Reflexive, discernible, symmetric, triangle, and non empty.

A metric space is a Reflexive discernible symmetric triangle metric structure.

Next we state four propositions:

(1) For every metric structure M holds for every element a of M holds $\rho(a,a) = 0$ iff M is Reflexive.

(2) Let M be a metric structure. Then for all elements a, b of M such that $\rho(a,b) = 0$ holds $a = b$ if and only if M is discernible.

(3) For every metric structure M holds for all elements a, b of M holds $\rho(a,b) = \rho(b,a)$ iff M is symmetric.

(4) For every metric structure M holds for all elements a, b, c of M holds $\rho(a,c) \leq \rho(a,b) + \rho(b,c)$ iff M is triangle.

Let M be a symmetric metric structure and let a, b be elements of M. Let us note that the functor $\rho(a,b)$ is commutative.

We now state two propositions:

(5) For every symmetric triangle Reflexive metric structure M and for all elements a, b of M holds $0 \leq \rho(a,b)$.

(6) Let M be a metric structure. Suppose that for all elements a, b, c of M holds $\rho(a,b) = 0$ iff $a = b$ and $\rho(a,b) = \rho(b,a)$ and $\rho(a,c) \leq \rho(a,b) + \rho(b,c)$. Then M is a metric space.

Let A be a set. The discrete metric of A yields a function from $[:A, A:]$ into $\mathbb{R}$ and is defined by the condition (Def. 11).

(Def. 11) Let x, y be elements of A. Then

(i) (the discrete metric of A)$(x, x) = 0$, and

(ii) if $x \neq y$, then (the discrete metric of A)$(x, y) = 1$.

Let A be a set. The discrete space on A yields a strict metric structure and is defined by:

(Def. 12) The discrete space on $A = \langle A,$ the discrete metric of $A\rangle$.

Let A be a non empty set. Note that the discrete space on A is non empty.

Let A be a set. Note that the discrete space on A is Reflexive, discernible, symmetric, and triangle.

The function $\rho_{\mathbb{R}}$ from $[:\mathbb{R}, \mathbb{R}:]$ into $\mathbb{R}$ is defined as follows:

(Def. 13) For all elements x, y of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x,y) = |x-y|$.

Next we state three propositions:

(9)[1] For all elements x, y of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x,y) = 0$ iff $x = y$.

(10) For all elements x, y of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x,y) = \rho_{\mathbb{R}}(y,x)$.

(11) For all elements x, y, z of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x,y) \leq \rho_{\mathbb{R}}(x,z) + \rho_{\mathbb{R}}(z,y)$.

The strict metric structure the metric space of real numbers is defined as follows:

(Def. 14) The metric space of real numbers $= \langle \mathbb{R}, \rho_{\mathbb{R}} \rangle$.

One can check that the metric space of real numbers is non empty.

Let us note that the metric space of real numbers is Reflexive, discernible, symmetric, and triangle.

Let M be a metric structure, let p be an element of M, and let r be a real number. The functor $\mathrm{Ball}(p,r)$ yields a subset of M and is defined by:

(Def. 15)(i) There exists a non empty metric structure M' and there exists an element p' of M' such that $M' = M$ and $p' = p$ and $\mathrm{Ball}(p,r) = \{q; q$ ranges over elements of M': $\rho(p',q) < r\}$ if M is non empty,

(ii) $\mathrm{Ball}(p,r)$ is empty, otherwise.

Let M be a metric structure, let p be an element of M, and let r be a real number. The functor $\overline{\mathrm{Ball}}(p,r)$ yields a subset of M and is defined by:

(Def. 16)(i) There exists a non empty metric structure M' and there exists an element p' of M' such that $M' = M$ and $p' = p$ and $\overline{\mathrm{Ball}}(p,r) = \{q; q$ ranges over elements of M': $\rho(p',q) \leq r\}$ if M is non empty,

(ii) $\overline{\mathrm{Ball}}(p,r)$ is empty, otherwise.

Let M be a metric structure, let p be an element of M, and let r be a real number. The functor $\mathrm{Sphere}(p,r)$ yields a subset of M and is defined by:

(Def. 17)(i) There exists a non empty metric structure M' and there exists an element p' of M' such that $M' = M$ and $p' = p$ and $\mathrm{Sphere}(p,r) = \{q; q$ ranges over elements of M': $\rho(p',q) = r\}$ if M is non empty,

(ii) $\mathrm{Sphere}(p,r)$ is empty, otherwise.

In the sequel r denotes a real number.

We now state several propositions:

(12) For every metric structure M and for all elements p, x of M holds $x \in \mathrm{Ball}(p,r)$ iff M is non empty and $\rho(p,x) < r$.

(13) For every metric structure M and for all elements p, x of M holds $x \in \overline{\mathrm{Ball}}(p,r)$ iff M is non empty and $\rho(p,x) \leq r$.

(14) For every metric structure M and for all elements p, x of M holds $x \in \mathrm{Sphere}(p,r)$ iff M is non empty and $\rho(p,x) = r$.

(15) For every metric structure M and for every element p of M holds $\mathrm{Ball}(p,r) \subseteq \overline{\mathrm{Ball}}(p,r)$.

(16) For every metric structure M and for every element p of M holds $\mathrm{Sphere}(p,r) \subseteq \overline{\mathrm{Ball}}(p,r)$.

(17) For every metric structure M and for every element p of M holds $\mathrm{Sphere}(p,r) \cup \mathrm{Ball}(p,r) = \overline{\mathrm{Ball}}(p,r)$.

[1] The propositions (7) and (8) have been removed.

(18) For every non empty metric structure M and for every element p of M holds $\mathrm{Ball}(p,r) = \{q; q$ ranges over elements of M: $\rho(p,q) < r\}$.

(19) For every non empty metric structure M and for every element p of M holds $\overline{\mathrm{Ball}}(p,r) = \{q; q$ ranges over elements of M: $\rho(p,q) \leq r\}$.

(20) For every non empty metric structure M and for every element p of M holds $\mathrm{Sphere}(p,r) = \{q; q$ ranges over elements of M: $\rho(p,q) = r\}$.

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