JOURNAL OF FORMALIZED MATHEMATICS
Volume 12, Released 2000, Published 2003
Inst. of Computer Science, Univ. of Białystok

The Measurability of Extended Real Valued Functions

Noboru Endou
Shinshu University
Nagano

Katsumi Wasaki
Shinshu University
Nagano

Yasunari Shidama
Shinshu University
Nagano

Summary. In this article we prove the measurablility of some extended real valued functions which are $f+g$, $f-g$ and so on. Moreover, we will define the simple function which are defined on the sigma field. It will play an important role for the Lebesgue integral theory.

MML Identifier: `MESFUNC2`.

WWW: `http://mizar.org/JFM/Vol12/mesfunc2.html`

The articles [20], [22], [1], [21], [17], [23], [11], [2], [18], [3], [4], [5], [10], [19], [6], [7], [8], [9], [12], [13], [14], [15], and [16] provide the notation and terminology for this paper.

1. Finite Valued Function

For simplicity, we follow the rules: X denotes a non empty set, x denotes an element of X, f, g denote partial functions from X to $\overline{\mathbb{R}}$, S denotes a σ-field of subsets of X, F denotes a function from $\mathbb{Q}$ into S, p denotes a rational number, r denotes a real number, n, m denote natural numbers, and A, B denote elements of S.

Let us consider X and let us consider f. We say that f is finite if and only if:

(Def. 1) For every x such that $x \in \operatorname{dom} f$ holds $|f(x)| < +\infty$.

We now state three propositions:

(1) $f = 1\, f$.

(2) For all f, g, A such that f is finite or g is finite holds $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $\operatorname{dom}(f-g) = \operatorname{dom} f \cap \operatorname{dom} g$.

(3) Let given f, g, F, r, A. Suppose f is finite and g is finite and for every p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r-p)))$. Then $A \cap \text{LE-dom}(f+g, \overline{\mathbb{R}}(r)) = \bigcup \operatorname{rng} F$.

2. Measurability of $f+g$ and $f-g$

We now state several propositions:

(4) There exists a function F from $\mathbb{N}$ into $\mathbb{Q}$ such that F is one-to-one and $\operatorname{dom} F = \mathbb{N}$ and $\operatorname{rng} F = \mathbb{Q}$.

(5) Let X, Y, Z be non empty sets and F be a function from X into Z. If $X \approx Y$, then there exists a function G from Y into Z such that $\operatorname{rng} F = \operatorname{rng} G$.

(6) Let given S, f, g, A. Suppose f is measurable on A and g is measurable on A. Then there exists a function F from $\mathbb{Q}$ into S such that for every rational number p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r-p)))$.

(7) Let given f, g, A. Suppose f is finite and g is finite and f is measurable on A and g is measurable on A. Then $f+g$ is measurable on A.

(9)[1] For every non empty set C and for all partial functions f_1, f_2 from C to $\overline{\mathbb{R}}$ holds $f_1 - f_2 = f_1 + -f_2$.

(10) For every real number r holds $\overline{\mathbb{R}}(-r) = -\overline{\mathbb{R}}(r)$.

(11) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $-f = (-1)\ f$.

(12) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and r be a real number. If f is finite, then $r\ f$ is finite.

(13) Let given f, g, A. Suppose f is finite and g is finite and f is measurable on A and g is measurable on A and $A \subseteq \text{dom}\, g$. Then $f - g$ is measurable on A.

3. Definitions of Extended Real Valued Functions $\max_+(f)$ and $\max_-(f)$ and their Basic Properties

Let C be a non empty set and let f be a partial function from C to $\overline{\mathbb{R}}$. The functor $\max_+(f)$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined by:

(Def. 2) $\text{dom} \max_+(f) = \text{dom} f$ and for every element x of C such that $x \in \text{dom} \max_+(f)$ holds $(\max_+(f))(x) = \max(f(x), 0_{\overline{\mathbb{R}}})$.

The functor $\max_-(f)$ yields a partial function from C to $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 3) $\text{dom} \max_-(f) = \text{dom} f$ and for every element x of C such that $x \in \text{dom} \max_-(f)$ holds $(\max_-(f))(x) = \max(-f(x), 0_{\overline{\mathbb{R}}})$.

One can prove the following propositions:

(14) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \text{dom} f$, then $0_{\overline{\mathbb{R}}} \leq (\max_+(f))(x)$.

(15) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \text{dom} f$, then $0_{\overline{\mathbb{R}}} \leq (\max_-(f))(x)$.

(16) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $\max_-(f) = \max_+(-f)$.

(17) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \text{dom} f$ and $0_{\overline{\mathbb{R}}} < (\max_+(f))(x)$, then $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$.

(18) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \text{dom} f$ and $0_{\overline{\mathbb{R}}} < (\max_-(f))(x)$, then $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$.

(19) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $\text{dom} f = \text{dom}(\max_+(f) - \max_-(f))$ and $\text{dom} f = \text{dom}(\max_+(f) + \max_-(f))$.

(20) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \text{dom} f$, then $(\max_+(f))(x) = f(x)$ or $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$ but $(\max_-(f))(x) = -f(x)$ or $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$.

(21) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \text{dom} f$ and $(\max_+(f))(x) = f(x)$, then $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$.

[1] The proposition (8) has been removed.

(22) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_{+}(f))(x) = 0_{\overline{\mathbb{R}}}$, then $(\max_{-}(f))(x) = -f(x)$.

(23) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_{-}(f))(x) = -f(x)$, then $(\max_{+}(f))(x) = 0_{\overline{\mathbb{R}}}$.

(24) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_{-}(f))(x) = 0_{\overline{\mathbb{R}}}$, then $(\max_{+}(f))(x) = f(x)$.

(25) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $f = \max_{+}(f) - \max_{-}(f)$.

(26) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $|f| = \max_{+}(f) + \max_{-}(f)$.

4. Measurability of $\max_{+}(f)$, $\max_{-}(f)$ and $|f|$

Next we state three propositions:

(27) If f is measurable on A, then $\max_{+}(f)$ is measurable on A.

(28) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $\max_{-}(f)$ is measurable on A.

(29) For all f, A such that f is measurable on A and $A \subseteq \operatorname{dom} f$ holds $|f|$ is measurable on A.

5. Definition and Measurability of Characteristic Function

Next we state the proposition

(30) For all sets A, X holds $\operatorname{rng}(\chi_{A,X}) \subseteq \{0_{\overline{\mathbb{R}}}, \overline{1}\}$.

Let A, X be sets. Then $\chi_{A,X}$ is a partial function from X to $\overline{\mathbb{R}}$.
One can prove the following two propositions:

(31) $\chi_{A,X}$ is finite.

(32) $\chi_{A,X}$ is measurable on B.

6. Definition and Measurability of Simple Function

Let X be a set and let S be a σ-field of subsets of X. Note that there exists a finite sequence of elements of S which is disjoint valued.

Let X be a set and let S be a σ-field of subsets of X. A finite sequence of separated subsets of S is a disjoint valued finite sequence of elements of S.

We now state two propositions:

(33) Suppose F is a finite sequence of separated subsets of S. Then there exists a sequence G of separated subsets of S such that $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} G$ and for every n such that $n \in \operatorname{dom} F$ holds $F(n) = G(n)$ and for every m such that $m \notin \operatorname{dom} F$ holds $G(m) = \emptyset$.

(34) If F is a finite sequence of separated subsets of S, then $\bigcup \operatorname{rng} F \in S$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is simple function in S if and only if the conditions (Def. 5) are satisfied.

(Def. 5)[2] (i) f is finite, and

(ii) there exists a finite sequence F of separated subsets of S such that $\operatorname{dom} f = \bigcup \operatorname{rng} F$ and for every natural number n and for all elements x, y of X such that $n \in \operatorname{dom} F$ and $x \in F(n)$ and $y \in F(n)$ holds $f(x) = f(y)$.

[2] The definition (Def. 4) has been removed.

One can prove the following propositions:

(35) If f is finite, then $\operatorname{rng} f$ is a subset of $\mathbb{R}$.

(36) Suppose F is a finite sequence of separated subsets of S. Let given n. Then $F \upharpoonright \operatorname{Seg} n$ is a finite sequence of separated subsets of S.

(37) If f is simple function in S, then f is measurable on A.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/wellord2.html.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[4] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/supinf_1.html.

[5] Józef Białas. Series of positive real numbers. Measure theory. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/supinf_2.html.

[6] Józef Białas. The σ-additive measure theory. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/measure1.html.

[7] Józef Białas. Several properties of the σ-additive measure. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/measure2.html.

[8] Józef Białas. Completeness of the σ-additive measure. Measure theory. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/measure3.html.

[9] Józef Białas. Some properties of the intervals. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/measure6.html.

[10] Czesław Byliński. Basic functions and operations on functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_3.html.

[11] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[12] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[13] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[14] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Journal of Formalized Mathematics*, 12, 2000. http://mizar.org/JFM/Vol12/extreal1.html.

[15] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Journal of Formalized Mathematics*, 12, 2000. http://mizar.org/JFM/Vol12/mesfunc1.html.

[16] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Some properties of extended real numbers operations: abs, min and max. *Journal of Formalized Mathematics*, 12, 2000. http://mizar.org/JFM/Vol12/extreal2.html.

[17] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[18] Andrzej Kondracki. Basic properties of rational numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rat_1.html.

[19] Andrzej Nędzusiak. Probability. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/prob_2.html.

[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[21] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[22] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[23] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received October 6, 2000

Published January 2, 2004
