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The σ-additive Measure Theory

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Summary. The article contains definition and basic properties of σ-additive, non-negative measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by [9]. We present definitions of σ-field of sets, σ-additive measure, measurable sets, measure zero sets and the basic theorems describing relationships between the notion mentioned above. The work is the third part of the series of articles concerning the Lebesgue measure theory.

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The articles [11], [10], [5], [14], [12], [15], [13], [3], [4], [8], [6], [7], [1], and [2] provide the notation and terminology for this paper.

In this paper X is a set.

We now state several propositions:

(1) For all sets X, Y holds $\bigcup\{X,Y,\emptyset\} = \bigcup\{X,Y\}$.

(4)[1] For all extended real numbers x, y, s, t such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq s$ and $x \leq y$ and $s \leq t$ holds $x+s \leq y+t$.

(5) For all extended real numbers x, y, z such that $0_{\overline{\mathbb{R}}} \leq y$ and $0_{\overline{\mathbb{R}}} \leq z$ and $x = y+z$ and $y < +\infty$ holds $z = x-y$.

(6) For every subset A of X holds $\{A\}$ is a family of subsets of X.

(7) For all subsets A, B of X holds $\{A,B\}$ is a family of subsets of X.

(8) For all subsets A, B, C of X holds $\{A,B,C\}$ is a non empty family of subsets of X.

(9) $\{\emptyset\}$ is a family of subsets of X.

The scheme *DomsetFamEx* deals with a set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a non empty family F of subsets of $\mathcal{A}$ such that for every set B holds $B \in F$ iff $B \subseteq \mathcal{A}$ and $\mathcal{P}[B]$

provided the parameters have the following property:

- There exists a set B such that $B \subseteq \mathcal{A}$ and $\mathcal{P}[B]$.

Let X be a set and let S be a non empty family of subsets of X. The functor $X \setminus S$ yielding a family of subsets of X is defined by:

(Def. 2)[2] For every set A holds $A \in X \setminus S$ iff there exists a set B such that $B \in S$ and $A = X \setminus B$.

[1] The propositions (2) and (3) have been removed.
[2] The definition (Def. 1) has been removed.

Let X be a set and let S be a non empty family of subsets of X. Observe that $X \setminus S$ is non empty. One can prove the following propositions:

(14)[3] For every non empty family S of subsets of X holds $S = X \setminus (X \setminus S)$.

(15) For every non empty family S of subsets of X holds $\bigcap S = X \setminus \bigcup (X \setminus S)$ and $\bigcup S = X \setminus \bigcap (X \setminus S)$.

Let X be a set and let I_1 be a family of subsets of X. Let us observe that I_1 is closed for complement operator if and only if:

(Def. 3) For every set A such that $A \in I_1$ holds $X \setminus A \in I_1$.

One can prove the following proposition

(16) Let X be a set and F be a family of subsets of X. Suppose F is $\cup$-closed and closed for complement operator. Then F is $\cap$-closed.

Let X be a set. Note that every family of subsets of X which is $\cup$-closed and closed for complement operator is also $\cap$-closed and every family of subsets of X which is $\cap$-closed and closed for complement operator is also $\cup$-closed.

The following propositions are true:

(17) For every field S of subsets of X holds $S = X \setminus S$.

(18) Let M be a set. Then M is a field of subsets of X if and only if there exists a non empty family S of subsets of X such that $M = S$ and for every set A such that $A \in S$ holds $X \setminus A \in S$ and for all sets A, B such that $A \in S$ and $B \in S$ holds $A \cup B \in S$.

(19) Let S be a non empty family of subsets of X. Then S is a field of subsets of X if and only if for every set A such that $A \in S$ holds $X \setminus A \in S$ and for all sets A, B such that $A \in S$ and $B \in S$ holds $A \cap B \in S$.

(20) For every field S of subsets of X and for all sets A, B such that $A \in S$ and $B \in S$ holds $A \setminus B \in S$.

(21) For every field S of subsets of X holds $\emptyset \in S$ and $X \in S$.

Let S be a non empty set, let F be a function from S into $\overline{\mathbb{R}}$, and let A be an element of S. Then $F(A)$ is an extended real number.

Let X be a non empty set and let F be a function from X into $\overline{\mathbb{R}}$. Let us observe that F is non-negative if and only if:

(Def. 4) For every element A of X holds $0_{\overline{\mathbb{R}}} \leq F(A)$.

Next we state the proposition

(23)[4] Let S be a field of subsets of X. Then there exists a function M from S into $\overline{\mathbb{R}}$ such that M is non-negative and $M(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for all elements A, B of S such that A misses B holds $M(A \cup B) = M(A) + M(B)$.

Let X be a set and let S be a field of subsets of X. A function from S into $\overline{\mathbb{R}}$ is said to be a measure on S if:

(Def. 5) It is non-negative and $\mathrm{it}(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for all elements A, B of S such that A misses B holds $\mathrm{it}(A \cup B) = \mathrm{it}(A) + \mathrm{it}(B)$.

Next we state two propositions:

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[3] The propositions (10)–(13) have been removed.

[4] The proposition (22) has been removed.

(25)[5] Let S be a field of subsets of X, M be a measure on S, and A, B be elements of S. If $A \subseteq B$, then $M(A) \leq M(B)$.

(26) Let S be a field of subsets of X, M be a measure on S, and A, B be elements of S. If $A \subseteq B$ and $M(A) < +\infty$, then $M(B \setminus A) = M(B) - M(A)$.

Let X be a set. Note that there exists a family of subsets of X which is non empty, closed for complement operator, and $\cap$-closed.

Let X be a set, let S be a non empty $\cup$-closed family of subsets of X, and let A, B be elements of S. Then $A \cup B$ is an element of S.

Let X be a set, let S be a field of subsets of X, and let A, B be elements of S. Then $A \cap B$ is an element of S. Then $A \setminus B$ is an element of S.

We now state the proposition

(27) For every field S of subsets of X and for every measure M on S and for all elements A, B of S holds $M(A \cup B) \leq M(A) + M(B)$.

Let X be a set, let S be a field of subsets of X, let M be a measure on S, and let A be a set. We say that A is measurable w.r.t. M if and only if:

(Def. 6) $A \in S$.

We now state the proposition

(29)[6] Let S be a field of subsets of X and M be a measure on S. Then

(i) $\emptyset$ is measurable w.r.t. M,

(ii) X is measurable w.r.t. M, and

(iii) for all sets A, B such that A is measurable w.r.t. M and B is measurable w.r.t. M holds $X \setminus A$ is measurable w.r.t. M and $A \cup B$ is measurable w.r.t. M and $A \cap B$ is measurable w.r.t. M.

Let X be a set, let S be a field of subsets of X, and let M be a measure on S. An element of S is called a set of measure zero w.r.t. M if:

(Def. 7) $M(\mathrm{it}) = 0_{\overline{\mathbb{R}}}$.

Next we state several propositions:

(31)[7] Let S be a field of subsets of X, M be a measure on S, A be an element of S, and B be a set of measure zero w.r.t. M. If $A \subseteq B$, then A is a set of measure zero w.r.t. M.

(32) Let S be a field of subsets of X, M be a measure on S, and A, B be sets of measure zero w.r.t. M. Then

(i) $A \cup B$ is a set of measure zero w.r.t. M,

(ii) $A \cap B$ is a set of measure zero w.r.t. M, and

(iii) $A \setminus B$ is a set of measure zero w.r.t. M.

(33) Let S be a field of subsets of X, M be a measure on S, A be an element of S, and B be a set of measure zero w.r.t. M. Then $M(A \cup B) = M(A)$ and $M(A \cap B) = 0_{\overline{\mathbb{R}}}$ and $M(A \setminus B) = M(A)$.

(34) For every subset A of X there exists a function F from $\mathbb{N}$ into 2^X such that $\mathrm{rng}\, F = \{A\}$.

(35) For every set A there exists a function F from $\mathbb{N}$ into $\{A\}$ such that for every natural number n holds $F(n) = A$.

[5] The proposition (24) has been removed.
[6] The proposition (28) has been removed.
[7] The proposition (30) has been removed.

Let X be a set. One can check that there exists a family of subsets of X which is non empty and countable.

Let X be a set. A denumerable family of subsets of X is a non empty countable family of subsets of X.

One can prove the following propositions:

(38)[8] Let A, B, C be subsets of X. Then there exists a function F from $\mathbb{N}$ into 2^X such that $\operatorname{rng} F = \{A, B, C\}$ and $F(0) = A$ and $F(1) = B$ and for every natural number n such that $1 < n$ holds $F(n) = C$.

(39) For all subsets A, B of X holds $\{A, B, \emptyset\}$ is a denumerable family of subsets of X.

(40) Let A, B be subsets of X. Then there exists a function F from $\mathbb{N}$ into 2^X such that $\operatorname{rng} F = \{A, B\}$ and $F(0) = A$ and for every natural number n such that $0 < n$ holds $F(n) = B$.

(41) For all subsets A, B of X holds $\{A, B\}$ is a denumerable family of subsets of X.

(42) For every denumerable family S of subsets of X holds $X \setminus S$ is a denumerable family of subsets of X.

Let X be a set and let I_1 be a family of subsets of X. We say that I_1 is σ-field of subsets-like if and only if:

(Def. 9)[9] For every denumerable family M of subsets of X such that $M \subseteq I_1$ holds $\bigcup M \in I_1$.

Let X be a set. One can verify that there exists a family of subsets of X which is non empty, closed for complement operator, and σ-field of subsets-like.

Let X be a set. A σ-field of subsets of X is a closed for complement operator σ-field of subsets-like non empty family of subsets of X.

We now state the proposition

(45)[10] For every σ-field S of subsets of X holds $\emptyset \in S$ and $X \in S$.

Let X be a set. One can check that every σ-field of subsets of X is non empty.

We now state four propositions:

(46) For every σ-field S of subsets of X and for all sets A, B such that $A \in S$ and $B \in S$ holds $A \cup B \in S$ and $A \cap B \in S$.

(47) For every σ-field S of subsets of X and for all sets A, B such that $A \in S$ and $B \in S$ holds $A \setminus B \in S$.

(48) For every σ-field S of subsets of X holds $S = X \setminus S$.

(49) Let S be a non empty family of subsets of X. Then for every set A such that $A \in S$ holds $X \setminus A \in S$ and for every denumerable family M of subsets of X such that $M \subseteq S$ holds $\bigcap M \in S$ if and only if S is a σ-field of subsets of X.

Let X be a set and let S be a σ-field of subsets of X. One can verify that there exists a function from $\mathbb{N}$ into S which is disjoint valued.

Let X be a set and let S be a σ-field of subsets of X. A sequence of separated subsets of S is a disjoint valued function from $\mathbb{N}$ into S.

Let X be a set, let S be a σ-field of subsets of X, and let F be a function from $\mathbb{N}$ into S. Then $\operatorname{rng} F$ is a non empty family of subsets of X.

The following propositions are true:

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[8] The propositions (36) and (37) have been removed.

[9] The definition (Def. 8) has been removed.

[10] The propositions (43) and (44) have been removed.

(52)[11] Let S be a σ-field of subsets of X and F be a function from $\mathbb{N}$ into S. Then $\operatorname{rng} F$ is a denumerable family of subsets of X.

(53) For every σ-field S of subsets of X and for every function F from $\mathbb{N}$ into S holds $\bigcup \operatorname{rng} F$ is an element of S.

(54) Let Y, S be non empty sets, F be a function from Y into S, and M be a function from S into $\overline{\mathbb{R}}$. If M is non-negative, then $M \cdot F$ is non-negative.

(55) Let S be a σ-field of subsets of X and a, b be extended real numbers. Then there exists a function M from S into $\overline{\mathbb{R}}$ such that for every element A of S holds

(i) if $A = \emptyset$, then $M(A) = a$, and

(ii) if $A \neq \emptyset$, then $M(A) = b$.

(56) Let S be a σ-field of subsets of X. Then there exists a function M from S into $\overline{\mathbb{R}}$ such that for every element A of S holds

(i) if $A = \emptyset$, then $M(A) = 0_{\overline{\mathbb{R}}}$, and

(ii) if $A \neq \emptyset$, then $M(A) = +\infty$.

(57) Let S be a σ-field of subsets of X. Then there exists a function M from S into $\overline{\mathbb{R}}$ such that for every element A of S holds $M(A) = 0_{\overline{\mathbb{R}}}$.

(58) Let S be a σ-field of subsets of X. Then there exists a function M from S into $\overline{\mathbb{R}}$ such that M is non-negative and $M(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for every sequence F of separated subsets of S holds $\sum(M \cdot F) = M(\bigcup \operatorname{rng} F)$.

Let X be a set and let S be a σ-field of subsets of X. A function from S into $\overline{\mathbb{R}}$ is said to be a σ-measure on S if:

(Def. 11)[12] It is non-negative and $\mathrm{it}(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for every sequence F of separated subsets of S holds $\sum(\mathrm{it} \cdot F) = \mathrm{it}(\bigcup \operatorname{rng} F)$.

Let X be a set. Note that every non empty family of subsets of X which is σ-field of subsets-like and closed for complement operator is also ∪-closed.

Next we state several propositions:

(60)[13] For every σ-field S of subsets of X holds every σ-measure on S is a measure on S.

(61) Let S be a σ-field of subsets of X, M be a σ-measure on S, and A, B be elements of S. If A misses B, then $M(A \cup B) = M(A) + M(B)$.

(62) Let S be a σ-field of subsets of X, M be a σ-measure on S, and A, B be elements of S. If $A \subseteq B$, then $M(A) \leq M(B)$.

(63) Let S be a σ-field of subsets of X, M be a σ-measure on S, and A, B be elements of S. If $A \subseteq B$ and $M(A) < +\infty$, then $M(B \setminus A) = M(B) - M(A)$.

(64) Let S be a σ-field of subsets of X, M be a σ-measure on S, and A, B be elements of S. Then $M(A \cup B) \leq M(A) + M(B)$.

Let X be a set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let A be a set. We say that A is measurable w.r.t. M if and only if:

(Def. 12) $A \in S$.

We now state two propositions:

[11] The propositions (50) and (51) have been removed.
[12] The definition (Def. 10) has been removed.
[13] The proposition (59) has been removed.

(66)[14] Let S be a σ-field of subsets of X and M be a σ-measure on S. Then

(i) $\emptyset$ is measurable w.r.t. M,

(ii) X is measurable w.r.t. M, and

(iii) for all sets A, B such that A is measurable w.r.t. M and B is measurable w.r.t. M holds $X \setminus A$ is measurable w.r.t. M and $A \cup B$ is measurable w.r.t. M and $A \cap B$ is measurable w.r.t. M.

(67) Let S be a σ-field of subsets of X, M be a σ-measure on S, and T be a denumerable family of subsets of X. Suppose that for every set A such that $A \in T$ holds A is measurable w.r.t. M. Then $\bigcup T$ is measurable w.r.t. M and $\bigcap T$ is measurable w.r.t. M.

Let X be a set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. An element of S is called a set of measure zero w.r.t. M if:

(Def. 13) $M(\mathrm{it}) = 0_{\overline{\mathbb{R}}}$.

One can prove the following propositions:

(69)[15] Let S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and B be a set of measure zero w.r.t. M. If $A \subseteq B$, then A is a set of measure zero w.r.t. M.

(70) Let S be a σ-field of subsets of X, M be a σ-measure on S, and A, B be sets of measure zero w.r.t. M. Then

(i) $A \cup B$ is a set of measure zero w.r.t. M,

(ii) $A \cap B$ is a set of measure zero w.r.t. M, and

(iii) $A \setminus B$ is a set of measure zero w.r.t. M.

(71) Let S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and B be a set of measure zero w.r.t. M. Then $M(A \cup B) = M(A)$ and $M(A \cap B) = 0_{\overline{\mathbb{R}}}$ and $M(A \setminus B) = M(A)$.

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