JOURNAL OF FORMALIZED MATHEMATICS
Volume 7, Released 1995, Published 2003
Inst. of Computer Science, Univ. of Białystok

Associated Matrix of Linear Map

Robert Milewski
Warsaw University
Białystok

MML Identifier: MATRLIN.

WWW: http://mizar.org/JFM/Vol7/matrlin.html

The articles [16], [7], [23], [2], [14], [3], [24], [4], [6], [5], [11], [12], [26], [25], [17], [13], [22], [19], [18], [10], [21], [15], [20], [1], [9], and [8] provide the notation and terminology for this paper.

1. Preliminaries

Let A be a set, let X be a set, let D be a non empty set of finite sequences of A, let p be a partial function from X to D, and let i be a set. Then p_i is an element of D.

We follow the rules: k, t, i, j, m, n are natural numbers, x is a set, and D is a non empty set.

Next we state the proposition

(2)[1] For every finite sequence p holds $\mathrm{rng}(p_{\restriction i}) \subseteq \mathrm{rng}\, p$.

Let D be a non empty set, let us consider k, and let M be a matrix over D. Then $M_{\restriction k}$ is a matrix over D.

We now state four propositions:

(3) For every finite sequence M such that $\mathrm{len}\, M = n+1$ holds $\mathrm{len}(M_{\restriction n+1}) = n$.

(4) Let M be a matrix over D of dimension $n+1 \times m$ and M_1 be a matrix over D. Then

(i) if $n > 0$, then $\mathrm{width}\, M = \mathrm{width}(M_{\restriction n+1})$, and

(ii) if $M_1 = \langle M(n+1) \rangle$, then $\mathrm{width}\, M = \mathrm{width}\, M_1$.

(5) For every matrix M over D of dimension $n+1 \times m$ holds $M_{\restriction n+1}$ is a matrix over D of dimension $n \times m$.

(6) For every finite sequence M such that $\mathrm{len}\, M = n+1$ holds $M = (M_{\restriction \mathrm{len}\, M}) ^\frown \langle M(\mathrm{len}\, M) \rangle$.

Let us consider D and let P be a finite sequence of elements of D. Then $\langle P \rangle$ is a matrix over D of dimension $1 \times \mathrm{len}\, P$.

2. More on Finite Sequence

We now state two propositions:

(7) For every set A and for every finite sequence F holds $(\mathrm{Sgm}(F^{-1}(A))) ^\frown \mathrm{Sgm}(F^{-1}(\mathrm{rng}\, F \setminus A))$ is a permutation of $\mathrm{dom}\, F$.

[1] The proposition (1) has been removed.

(8) Let F be a finite sequence and A be a subset of $\operatorname{rng} F$. Suppose F is one-to-one. Then there exists a permutation p of $\operatorname{dom} F$ such that $(F - A^{c}) \frown (F - A) = F \cdot p$.

Let I_1 be a function. We say that I_1 is finite sequence yielding if and only if:

(Def. 1) For every x such that $x \in \operatorname{dom} I_1$ holds $I_1(x)$ is a finite sequence.

One can verify that there exists a function which is finite sequence yielding.

Let F, G be finite sequence yielding functions. The functor $F \frown G$ yields a finite sequence yielding function and is defined by the conditions (Def. 2).

(Def. 2)(i) $\operatorname{dom}(F \frown G) = \operatorname{dom} F \cap \operatorname{dom} G$, and

(ii) for every set i such that $i \in \operatorname{dom}(F \frown G)$ and for all finite sequences f, g such that $f = F(i)$ and $g = G(i)$ holds $(F \frown G)(i) = f \frown g$.

3. Matrices and Finite Sequences in Vector Space

For simplicity, we use the following convention: K is a field, V is a vector space over K, a is an element of K, W is an element of V, K_1, K_2, K_3 are linear combinations of V, and X is a subset of V.

One can prove the following four propositions:

(9) If X is linearly independent and the support of $K_1 \subseteq X$ and the support of $K_2 \subseteq X$ and $\sum K_1 = \sum K_2$, then $K_1 = K_2$.

(10) Suppose that

(i) X is linearly independent,

(ii) the support of $K_1 \subseteq X$,

(iii) the support of $K_2 \subseteq X$,

(iv) the support of $K_3 \subseteq X$, and

(v) $\sum K_1 = \sum K_2 + \sum K_3$.

Then $K_1 = K_2 + K_3$.

(11) Suppose X is linearly independent and the support of $K_1 \subseteq X$ and the support of $K_2 \subseteq X$ and $a \neq 0_K$ and $\sum K_1 = a \cdot \sum K_2$. Then $K_1 = a \cdot K_2$.

(12) For every basis b_2 of V there exists a linear combination K_4 of V such that $W = \sum K_4$ and the support of $K_4 \subseteq b_2$.

Let K be a field and let V be a vector space over K. We say that V is finite dimensional if and only if:

(Def. 3) There exists a finite subset of V which is a basis of V.

Let K be a field. Observe that there exists a vector space over K which is strict and finite dimensional.

Let K be a field and let V be a finite dimensional vector space over K. A finite sequence of elements of the carrier of V is said to be an ordered basis of V if:

(Def. 4) It is one-to-one and rng it is a basis of V.

For simplicity, we adopt the following rules: p is a finite sequence, V_1, V_2, V_3 are finite dimensional vector spaces over K, f, f_1, f_2 are maps from V_1 into V_2, g is a map from V_2 into V_3, b_1 is an ordered basis of V_1, b_2 is an ordered basis of V_2, b_3 is an ordered basis of V_3, v_1, v_2 are vectors of V_2, v is an element of V_1, p_2, F are finite sequences of elements of the carrier of V_1, p_1, d are finite sequences of elements of the carrier of K, and K_4 is a linear combination of V_1.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V_1, V_2 be vector spaces over K, and let f_1, f_2 be maps from V_1 into V_2. The functor $f_1 + f_2$ yielding a map from V_1 into V_2 is defined as follows:

(Def. 5) For every element v of V_1 holds $(f_1+f_2)(v) = f_1(v)+f_2(v)$.

Let us consider K, let us consider V_1, V_2, let us consider f, and let a be an element of K. The functor $a \cdot f$ yields a map from V_1 into V_2 and is defined by:

(Def. 6) For every element v of V_1 holds $(a \cdot f)(v) = a \cdot f(v)$.

Next we state three propositions:

(13) Let a be an element of V_1, F be a finite sequence of elements of the carrier of V_1, and G be a finite sequence of elements of the carrier of K. Suppose $\operatorname{len} F = \operatorname{len} G$ and for every k and for every element v of K such that $k \in \operatorname{dom} F$ and $v = G(k)$ holds $F(k) = v \cdot a$. Then $\sum F = \sum G \cdot a$.

(14) Let a be an element of V_1, F be a finite sequence of elements of the carrier of K, and G be a finite sequence of elements of the carrier of V_1. If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{dom} F$ holds $G(k) = F_k \cdot a$, then $\sum G = \sum F \cdot a$.

(15) Let V_1 be an add-associative right zeroed right complementable non empty loop structure and F be a finite sequence of elements of the carrier of V_1. If for every k such that $k \in \operatorname{dom} F$ holds $F_k = 0_{(V_1)}$, then $\sum F = 0_{(V_1)}$.

Let us consider K, let us consider V_1, and let us consider p_1, p_2. The functor $\operatorname{lmlt}(p_1, p_2)$ yields a finite sequence of elements of the carrier of V_1 and is defined by:

(Def. 7) $\operatorname{lmlt}(p_1, p_2) = (\text{the left multiplication of } V_1)^{\circ}(p_1, p_2)$.

Next we state the proposition

(16) If $\operatorname{dom} p_1 = \operatorname{dom} p_2$, then $\operatorname{dom} \operatorname{lmlt}(p_1, p_2) = \operatorname{dom} p_1$.

Let V_1 be a non empty loop structure and let M be a finite sequence of elements of (the carrier of $V_1)^*$. The functor $\sum M$ yielding a finite sequence of elements of the carrier of V_1 is defined by:

(Def. 8) $\operatorname{len} \sum M = \operatorname{len} M$ and for every k such that $k \in \operatorname{dom} \sum M$ holds $(\sum M)_k = \sum (M_k)$.

One can prove the following propositions:

(17) For every matrix M over the carrier of V_1 such that $\operatorname{len} M = 0$ holds $\sum \sum M = 0_{(V_1)}$.

(18) For every matrix M over the carrier of V_1 of dimension $m+1 \times 0$ holds $\sum \sum M = 0_{(V_1)}$.

(19) For every element x of V_1 holds $\langle\langle x \rangle\rangle = \langle\langle x \rangle\rangle^{\mathrm{T}}$.

(20) For every finite sequence p of elements of the carrier of V_1 such that f is linear holds $f(\sum p) = \sum (f \cdot p)$.

(21) Let a be a finite sequence of elements of the carrier of K and p be a finite sequence of elements of the carrier of V_1. If $\operatorname{len} p = \operatorname{len} a$, then if f is linear, then $f \cdot \operatorname{lmlt}(a, p) = \operatorname{lmlt}(a, f \cdot p)$.

(22) Let a be a finite sequence of elements of the carrier of K. If $\operatorname{len} a = \operatorname{len} b_2$, then if g is linear, then $g(\sum \operatorname{lmlt}(a, b_2)) = \sum \operatorname{lmlt}(a, g \cdot b_2)$.

(23) Let F, F_1 be finite sequences of elements of the carrier of V_1, K_4 be a linear combination of V_1, and p be a permutation of $\operatorname{dom} F$. If $F_1 = F \cdot p$, then $K_4\, F_1 = (K_4\, F) \cdot p$.

(24) If F is one-to-one and the support of $K_4 \subseteq \operatorname{rng} F$, then $\sum (K_4\, F) = \sum K_4$.

(25) Let A be a set and p be a finite sequence of elements of the carrier of V_1. Suppose $\operatorname{rng} p \subseteq A$. Suppose f_1 is linear and f_2 is linear and for every v such that $v \in A$ holds $f_1(v) = f_2(v)$. Then $f_1(\sum p) = f_2(\sum p)$.

(26) If f_1 is linear and f_2 is linear, then for every ordered basis b_1 of V_1 such that $\operatorname{len} b_1 > 0$ holds if $f_1 \cdot b_1 = f_2 \cdot b_1$, then $f_1 = f_2$.

Let D be a non empty set. One can check that every matrix over D is finite sequence yielding.

Let D be a non empty set and let F, G be matrices over D. Then $F \frown G$ is a matrix over D.

Let D be a non empty set, let us consider n, m, k, let M_1 be a matrix over D of dimension $n \times k$, and let M_2 be a matrix over D of dimension $m \times k$. Then $M_1 \frown M_2$ is a matrix over D of dimension $n+m \times k$.

One can prove the following propositions:

(27) Let M_1 be a matrix over D of dimension $n \times k$ and M_2 be a matrix over D of dimension $m \times k$. If $i \in \operatorname{dom} M_1$, then $\operatorname{Line}(M_1 \frown M_2, i) = \operatorname{Line}(M_1, i)$.

(28) Let M_1 be a matrix over D of dimension $n \times k$ and M_2 be a matrix over D of dimension $m \times k$. If $\operatorname{width} M_1 = \operatorname{width} M_2$, then $\operatorname{width}(M_1 \frown M_2) = \operatorname{width} M_1$.

(29) Let M_1 be a matrix over D of dimension $t \times k$ and M_2 be a matrix over D of dimension $m \times k$. If $n \in \operatorname{dom} M_2$ and $i = \operatorname{len} M_1 + n$, then $\operatorname{Line}(M_1 \frown M_2, i) = \operatorname{Line}(M_2, n)$.

(30) Let M_1 be a matrix over D of dimension $n \times k$ and M_2 be a matrix over D of dimension $m \times k$. If $\operatorname{width} M_1 = \operatorname{width} M_2$, then for every i such that $i \in \operatorname{Seg} \operatorname{width} M_1$ holds $(M_1 \frown M_2)_{\square,i} = ((M_1)_{\square,i}) \frown ((M_2)_{\square,i})$.

(31) Let M_1 be a matrix over the carrier of V of dimension $n \times k$ and M_2 be a matrix over the carrier of V of dimension $m \times k$. Then $\sum(M_1 \frown M_2) = (\sum M_1) \frown \sum M_2$.

(32) Let M_1 be a matrix over D of dimension $n \times k$ and M_2 be a matrix over D of dimension $m \times k$. If $\operatorname{width} M_1 = \operatorname{width} M_2$, then $(M_1 \frown M_2)^{\mathrm{T}} = ({M_1}^{\mathrm{T}}) \frown {M_2}^{\mathrm{T}}$.

(33) For all matrices M_1, M_2 over the carrier of V_1 holds (the addition of $V_1)^{\circ}(\sum M_1, \sum M_2) = \sum(M_1 \frown M_2)$.

Let D be a non empty set, let F be a binary operation on D, and let P_1, P_2 be finite sequences of elements of D. Then $F^{\circ}(P_1, P_2)$ is a finite sequence of elements of D.

We now state several propositions:

(34) Let P_1, P_2 be finite sequences of elements of the carrier of V_1. If $\operatorname{len} P_1 = \operatorname{len} P_2$, then $\sum(($the addition of $V_1)^{\circ}(P_1, P_2)) = \sum P_1 + \sum P_2$.

(35) For all matrices M_1, M_2 over the carrier of V_1 such that $\operatorname{len} M_1 = \operatorname{len} M_2$ holds $\sum\sum M_1 + \sum\sum M_2 = \sum\sum(M_1 \frown M_2)$.

(36) For every finite sequence P of elements of the carrier of V_1 holds $\sum\sum\langle P\rangle = \sum\sum(\langle P\rangle^{\mathrm{T}})$.

(37) For every matrix M over the carrier of V_1 such that $\operatorname{len} M = n$ holds $\sum\sum M = \sum\sum(M^{\mathrm{T}})$.

(38) Let M be a matrix over the carrier of K of dimension $n \times m$. Suppose $n > 0$ and $m > 0$. Let p, d be finite sequences of elements of the carrier of K. Suppose $\operatorname{len} p = n$ and $\operatorname{len} d = m$ and for every j such that $j \in \operatorname{dom} d$ holds $d_j = \sum(p \bullet M_{\square,j})$. Let b, c be finite sequences of elements of the carrier of V_1. Suppose $\operatorname{len} b = m$ and $\operatorname{len} c = n$ and for every i such that $i \in \operatorname{dom} c$ holds $c_i = \sum \operatorname{lmlt}(\operatorname{Line}(M, i), b)$. Then $\sum \operatorname{lmlt}(p, c) = \sum \operatorname{lmlt}(d, b)$.

4. Decomposition of a Vector in Basis

Let K be a field, let V be a finite dimensional vector space over K, let b_1 be an ordered basis of V, and let W be an element of V. The functor $W \rightarrow b_1$ yields a finite sequence of elements of the carrier of K and is defined by the conditions (Def. 9).

(Def. 9)(i) $\operatorname{len}(W \rightarrow b_1) = \operatorname{len} b_1$, and

(ii) there exists a linear combination K_4 of V such that $W = \sum K_4$ and the support of $K_4 \subseteq \operatorname{rng} b_1$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len}(W \rightarrow b_1)$ holds $(W \rightarrow b_1)_k = K_4((b_1)_k)$.

The following four propositions are true:

(39) If $v_1 \rightarrow b_2 = v_2 \rightarrow b_2$, then $v_1 = v_2$.

(40) $v = \sum \mathrm{lmlt}(v \rightarrow b_1, b_1)$.

(41) If $\mathrm{len}\, d = \mathrm{len}\, b_1$, then $d = \sum \mathrm{lmlt}(d, b_1) \rightarrow b_1$.

(42) Let a, d be finite sequences of elements of the carrier of K. Suppose $\mathrm{len}\, a = \mathrm{len}\, b_2$. Let j be a natural number. Suppose $j \in \mathrm{dom}\, b_3$ and $\mathrm{len}\, d = \mathrm{len}\, b_2$ and for every k such that $k \in \mathrm{dom}\, b_2$ holds $d(k) = (g((b_2)_k) \rightarrow b_3)_j$. If $\mathrm{len}\, b_2 > 0$, then $(\sum \mathrm{lmlt}(a, g \cdot b_2) \rightarrow b_3)_j = \sum (a \bullet d)$.

5. Associated Matrix of Linear Map

Let K be a field, let V_1, V_2 be finite dimensional vector spaces over K, let f be a function from the carrier of V_1 into the carrier of V_2, let b_1 be a finite sequence of elements of the carrier of V_1, and let b_2 be an ordered basis of V_2. The functor $\mathrm{AutMt}(f, b_1, b_2)$ yields a matrix over K and is defined by:

(Def. 10) $\mathrm{len}\,\mathrm{AutMt}(f, b_1, b_2) = \mathrm{len}\, b_1$ and for every k such that $k \in \mathrm{dom}\, b_1$ holds $(\mathrm{AutMt}(f, b_1, b_2))_k = f((b_1)_k) \rightarrow b_2$.

We now state several propositions:

(43) If $\mathrm{len}\, b_1 = 0$, then $\mathrm{AutMt}(f, b_1, b_2) = \emptyset$.

(44) If $\mathrm{len}\, b_1 > 0$, then $\mathrm{width}\,\mathrm{AutMt}(f, b_1, b_2) = \mathrm{len}\, b_2$.

(45) If f_1 is linear and f_2 is linear and $\mathrm{AutMt}(f_1, b_1, b_2) = \mathrm{AutMt}(f_2, b_1, b_2)$ and $\mathrm{len}\, b_1 > 0$, then $f_1 = f_2$.

(46) If f is linear and g is linear and $\mathrm{len}\, b_1 > 0$ and $\mathrm{len}\, b_2 > 0$ and $\mathrm{len}\, b_3 > 0$, then $\mathrm{AutMt}(g \cdot f, b_1, b_3) = \mathrm{AutMt}(f, b_1, b_2) \cdot \mathrm{AutMt}(g, b_2, b_3)$.

(47) $\mathrm{AutMt}(f_1 + f_2, b_1, b_2) = \mathrm{AutMt}(f_1, b_1, b_2) + \mathrm{AutMt}(f_2, b_1, b_2)$.

(48) If $a \neq 0_K$, then $\mathrm{AutMt}(a \cdot f, b_1, b_2) = a \cdot \mathrm{AutMt}(f, b_1, b_2)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[8] Czesław Byliński. Binary operations applied to finite sequences. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseqop.html`.

[9] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_2.html`.

[10] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[11] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/matrix_1.html`.

[12] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/matrix_2.html`.

[13] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[14] Michał Muzalewski. Rings and modules — part II. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/mod_2.html.

[15] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[17] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[18] Wojciech A. Trybulec. Basis of vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_7.html.

[19] Wojciech A. Trybulec. Linear combinations in vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_6.html.

[20] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_3.html.

[21] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[22] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_4.html.

[23] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[24] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[25] Katarzyna Zawadzka. Sum and product of finite sequences of elements of a field. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/fvsum_1.html.

[26] Katarzyna Zawadzka. The product and the determinant of matrices with entries in a field. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/matrix_3.html.

Received June 30, 1995

Published January 2, 2004