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Transpose Matrices and Groups of Permutations

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Summary. Some facts concerning matrices with dimension 2×2 are shown. Upper and lower triangular matrices, and operation of deleting rows and columns in a matrix are introduced. Besides, we deal with sets of permutations and the fact that all permutations of finite set constitute a finite group is proved. Some proofs are based on [11] and [14].

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The articles [15], [8], [20], [21], [5], [7], [6], [2], [18], [19], [4], [17], [13], [3], [1], [12], [10], [16], and [9] provide the notation and terminology for this paper.

1. Some examples of matrices

For simplicity, we follow the rules: x, x_1, x_2, y_1, y_2 are sets, i, j, k, l, n, m are natural numbers, D is a non empty set, K is a field, s is a finite sequence, and a, b, c, d are elements of D.

The scheme *SeqDEx* deals with a non empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

> There exists a finite sequence p of elements of $\mathcal{A}$ such that $\operatorname{dom} p = \operatorname{Seg} \mathcal{B}$ and for every k such that $k \in \operatorname{Seg} \mathcal{B}$ holds $\mathcal{P}[k, p(k)]$

provided the parameters satisfy the following condition:

- For every k such that $k \in \operatorname{Seg} \mathcal{B}$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[k,x]$.

Let us consider n, m and let a be a set. The functor $\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times m}$ yields a tabular finite sequence and is defined by:

(Def. 1) $$\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times m} = n \mapsto (m \mapsto a).$$

Let us consider D, n, m and let us consider d. Then $\begin{pmatrix} d & \ldots & d \\ \vdots & \ddots & \vdots \\ d & \ldots & d \end{pmatrix}^{n \times m}$ is a matrix over D of dimension $n \times m$.

The following proposition is true

(1) If $\langle i, j \rangle \in$ the indices of $\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times m}$, then $\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times m} \circ (i, j) = a$.

In the sequel a', b' are elements of K.

Next we state the proposition

(2) $$\begin{pmatrix} a' & \dots & a' \\ \vdots & \ddots & \vdots \\ a' & \dots & a' \end{pmatrix}^{n \times n} + \begin{pmatrix} b' & \dots & b' \\ \vdots & \ddots & \vdots \\ b' & \dots & b' \end{pmatrix}^{n \times n} = \begin{pmatrix} a'+b' & \dots & a'+b' \\ \vdots & \ddots & \vdots \\ a'+b' & \dots & a'+b' \end{pmatrix}^{n \times n}.$$

Let a, b, c, d be sets. The functor $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ yielding a tabular finite sequence is defined as follows:

(Def. 2) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \langle \langle a, b \rangle, \langle c, d \rangle \rangle$.

Next we state two propositions:

(3) $\operatorname{len} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = 2$ and $\operatorname{width} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = 2$ and the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = [: \operatorname{Seg} 2, \operatorname{Seg} 2 :]$.

(4)(i) $\langle 1, 1 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$,

(ii) $\langle 1, 2 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$,

(iii) $\langle 2, 1 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$, and

(iv) $\langle 2, 2 \rangle \in$ the indices of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$.

Let us consider D and let a be an element of D. Then $\langle a \rangle$ is an element of D^1.

Let us consider D, let us consider n, and let p be an element of D^n. Then $\langle p \rangle$ is a matrix over D of dimension $1 \times n$.

The following proposition is true

(5) $\langle 1, 1 \rangle \in$ the indices of $\langle \langle a \rangle \rangle$ and $\langle \langle a \rangle \rangle \circ (1, 1) = a$.

Let us consider D and let a, b, c, d be elements of D. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix over D of dimension 2.

The following proposition is true

(6) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (1, 1) = a$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (1, 2) = b$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (2, 1) = c$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (2, 2) = d$.

Let us consider n and let K be a field. A matrix over K of dimension n is said to be an upper triangular matrix over K of dimension n if:

(Def. 3) For all i, j such that $\langle i, j \rangle \in$ the indices of it holds if $i > j$, then it $\circ (i, j) = 0_K$.

Let us consider n and let us consider K. A matrix over K of dimension n is said to be a lower triangular matrix over K of dimension n if:

(Def. 4) For all i, j such that $\langle i, j \rangle \in$ the indices of it holds if $i < j$, then it $\circ (i, j) = 0_K$.

We now state the proposition

(7) For every matrix M over D such that $\operatorname{len} M = n$ holds M is a matrix over D of dimension $n \times \operatorname{width} M$.

2. DELETING OF ROWS AND COLUMNS IN A MATRIX

Let us consider i and let p be a finite sequence. The functor $p_{\restriction i}$ yields a finite sequence and is defined as follows:

(Def. 5) $p_{\restriction i} = p \cdot \mathrm{Sgm}(\mathrm{dom}\, p \setminus \{i\})$.

One can prove the following three propositions:

(8) For every finite sequence p holds if $i \in \mathrm{dom}\, p$, then there exists m such that $\mathrm{len}\, p = m+1$ and $\mathrm{len}(p_{\restriction i}) = m$ and if $i \notin \mathrm{dom}\, p$, then $p_{\restriction i} = p$.

(9) For every finite sequence p of elements of D holds $p_{\restriction i}$ is a finite sequence of elements of D.

(10) For every matrix M over K of dimension $n \times m$ and for every k such that $k \in \mathrm{Seg}\, n$ holds $M(k) = \mathrm{Line}(M, k)$.

Let us consider i, let us consider K, and let M be a matrix over K. Let us assume that $i \in \mathrm{Seg}\,\mathrm{width}\, M$. The deleting of i-column in M yielding a matrix over K is defined by the conditions (Def. 6).

(Def. 6)(i) $\mathrm{len}(\text{the deleting of } i\text{-column in } M) = \mathrm{len}\, M$, and

(ii) for every k such that $k \in \mathrm{dom}\, M$ holds $(\text{the deleting of } i\text{-column in } M)(k) = \mathrm{Line}(M, k)_{\restriction i}$.

One can prove the following propositions:

(11) For all matrices M_1, M_2 over D such that $M_1{}^{\mathrm{T}} = M_2{}^{\mathrm{T}}$ and $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ holds $M_1 = M_2$.

(12) For every matrix M over D such that $\mathrm{width}\, M > 0$ holds $\mathrm{len}(M^{\mathrm{T}}) = \mathrm{width}\, M$ and $\mathrm{width}(M^{\mathrm{T}}) = \mathrm{len}\, M$.

(13) For all matrices M_1, M_2 over D such that $\mathrm{width}\, M_1 > 0$ and $\mathrm{width}\, M_2 > 0$ and $M_1{}^{\mathrm{T}} = M_2{}^{\mathrm{T}}$ and $\mathrm{width}(M_1{}^{\mathrm{T}}) = \mathrm{width}(M_2{}^{\mathrm{T}})$ holds $M_1 = M_2$.

(14) For all matrices M_1, M_2 over D such that $\mathrm{width}\, M_1 > 0$ and $\mathrm{width}\, M_2 > 0$ holds $M_1 = M_2$ iff $M_1{}^{\mathrm{T}} = M_2{}^{\mathrm{T}}$ and $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$.

(15) For every matrix M over D such that $\mathrm{len}\, M > 0$ and $\mathrm{width}\, M > 0$ holds $(M^{\mathrm{T}})^{\mathrm{T}} = M$.

(16) For every matrix M over D and for every i such that $i \in \mathrm{dom}\, M$ holds $\mathrm{Line}(M, i) = (M^{\mathrm{T}})_{\square, i}$.

(17) For every matrix M over D and for every j such that $j \in \mathrm{Seg}\,\mathrm{width}\, M$ holds $\mathrm{Line}(M^{\mathrm{T}}, j) = M_{\square, j}$.

(18) For every matrix M over D and for every i such that $i \in \mathrm{dom}\, M$ holds $M(i) = \mathrm{Line}(M, i)$.

Let us consider i, let us consider K, and let M be a matrix over K. Let us assume that $i \in \mathrm{dom}\, M$ and $\mathrm{width}\, M > 0$. The deleting of i-row in M yields a matrix over K and is defined as follows:

(Def. 7)(i) The deleting of i-row in $M = \emptyset$ if $\mathrm{len}\, M = 1$,

(ii) $\mathrm{width}(\text{the deleting of } i\text{-row in } M) = \mathrm{width}\, M$ and for every k such that $k \in \mathrm{Seg}\,\mathrm{width}\, M$ holds $(\text{the deleting of } i\text{-row in } M)_{\square, k} = (M_{\square, k})_{\restriction i}$, otherwise.

Let us consider i, j, let us consider n, let us consider K, and let M be a matrix over K of dimension n. The deleting of i-row and j-column in M yields a matrix over K and is defined by:

(Def. 8) $$\text{The deleting of } i\text{-row and } j\text{-column in } M = \begin{cases} \text{(i)} \quad \emptyset, \text{ if } n = 1, \\ \text{the deleting of } j\text{-column in the deleting of } i\text{-row in } M, \text{ otherwise.} \end{cases}$$

3. Sets of permutations

Let I_1 be a set. We say that I_1 is permutational if and only if:

(Def. 9) There exists n such that for every x such that $x \in I_1$ holds x is a permutation of $\mathrm{Seg}\,n$.

Let us observe that there exists a set which is permutational and non empty.

Let P be a permutational non empty set. The functor $\mathrm{len}\,P$ yielding a natural number is defined as follows:

(Def. 10) There exists s such that $s \in P$ and $\mathrm{len}\,P = \mathrm{len}\,s$.

Let P be a permutational non empty set. We see that the element of P is a permutation of $\mathrm{Seg}\,\mathrm{len}\,P$.

One can prove the following proposition

(19) There exists a permutational non empty set P such that $\mathrm{len}\,P = n$.

Let us consider n. The permutations of n-element set constitute a set defined by:

(Def. 11) $x \in$ the permutations of n-element set iff x is a permutation of $\mathrm{Seg}\,n$.

Let us consider n. One can check that the permutations of n-element set is permutational and non empty.

We now state two propositions:

(20) len(the permutations of n-element set) $= n$.

(21) The permutations of 1-element set $= \{\mathrm{idseq}(1)\}$.

Let us consider n and let p be an element of the permutations of n-element set. The functor $\mathrm{len}\,p$ yielding a natural number is defined as follows:

(Def. 12) There exists a finite sequence s such that $s = p$ and $\mathrm{len}\,p = \mathrm{len}\,s$.

We now state the proposition

(22) For every element p of the permutations of n-element set holds $\mathrm{len}\,p = n$.

4. Group of permutations

In the sequel p, q are elements of the permutations of n-element set.

Let us consider n. The functor A_n yielding a strict groupoid is defined by the conditions (Def. 13).

(Def. 13)(i) The carrier of $A_n =$ the permutations of n-element set, and

(ii) for all elements q, p of the permutations of n-element set holds (the multiplication of $A_n)(q, p) = p \cdot q$.

Let us consider n. Note that A_n is non empty.

The following propositions are true:

(23) $\mathrm{idseq}(n)$ is an element of A_n.

(24) $p \cdot \mathrm{idseq}(n) = p$ and $\mathrm{idseq}(n) \cdot p = p$.

(25) $p \cdot p^{-1} = \mathrm{idseq}(n)$ and $p^{-1} \cdot p = \mathrm{idseq}(n)$.

(26) p^{-1} is an element of A_n.

Let us consider n. A permutation of n element set is an element of the permutations of n-element set. Note that A_n is associative and group-like.

The following proposition is true

(28)[1] $\text{idseq}(n) = 1_{A_n}$.

Let us consider n and let p be a permutation of $\text{Seg}\, n$. We say that p is transposition if and only if:

(Def. 14) There exist i, j such that $i \in \text{dom}\, p$ and $j \in \text{dom}\, p$ and $i \neq j$ and $p(i) = j$ and $p(j) = i$ and for every k such that $k \neq i$ and $k \neq j$ and $k \in \text{dom}\, p$ holds $p(k) = k$.

We introduce p is a transposition as a synonym of p is transposition.

Let us consider n and let I_1 be a permutation of $\text{Seg}\, n$. We say that I_1 is even if and only if the condition (Def. 15) is satisfied.

(Def. 15) There exists a finite sequence l of elements of the carrier of A_n such that $\text{len}\, l \bmod 2 = 0$ and $I_1 = \prod l$ and for every i such that $i \in \text{dom}\, l$ there exists q such that $l(i) = q$ and q is a transposition.

We introduce I_1 is odd as an antonym of I_1 is even.

The following proposition is true

(29) $\text{id}_{\text{Seg}\, n}$ is even.

Let us consider K, n, let x be an element of K, and let p be an element of the permutations of n-element set. The functor $(-1)^{\text{sgn}(p)}x$ yields an element of K and is defined as follows:

(Def. 16) $(-1)^{\text{sgn}(p)}x = \begin{cases} \text{(i)}\quad x, & \text{if } p \text{is even}, \\ -x, & \text{otherwise}. \end{cases}$

Let X be a set. Let us assume that X is finite. The functor Ω^{f}_X yields an element of $\text{Fin}\, X$ and is defined as follows:

(Def. 17) $\Omega^{\text{f}}_X = X$.

One can prove the following proposition

(30) The permutations of n-element set are finite.

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