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Submodules

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Summary. This article contains the notions of trivial and non-trivial leftmodules and rings, cyclic submodules and inclusion of submodules. A few basic theorems related to these notions are proved.

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The articles [6], [2], [11], [12], [1], [7], [3], [10], [9], [8], [4], and [5] provide the notation and terminology for this paper.

1. PRELIMINARIES

For simplicity, we use the following convention: x is a set, K is a ring, r is a scalar of K, V, M, M_1, M_2, N are left modules over K, a is a vector of V, m, m_1, m_2 are vectors of M, n, n_1, n_2 are vectors of N, A is a subset of V, l is a linear combination of A, and W, W_1, W_2, W_3 are subspaces of V.

Let us consider K, V. We introduce $\mathrm{Sub}(V)$ as a synonym of Subspaces$V$.

The following four propositions are true:

(1) If $M_1 =$ the vector space structure of M_2, then $x \in M_1$ iff $x \in M_2$.

(2) For every vector v of the vector space structure of V such that $a = v$ holds $r \cdot a = r \cdot v$.

(3) The vector space structure of V is a strict subspace of V.

(4) V is a subspace of Ω_V.

2. TRIVIAL AND NON-TRIVIAL MODULES AND RINGS

Let us consider K. Let us observe that K is trivial if and only if:

(Def. 2)[1] $0_K = \mathbf{1}_K$.

Next we state three propositions:

(5) If K is trivial, then for every r holds $r = 0_K$ and for every a holds $a = 0_V$.

(6) If K is trivial, then V is trivial.

(7) V is trivial iff the vector space structure of $V = \mathbf{0}_V$.

[1] The definition (Def. 1) has been removed.

3. SUBMODULES AND SUBSETS

Let us consider K, V and let W be a strict subspace of V. The functor ${}^{@}W$ yielding an element of $\mathrm{Sub}(V)$ is defined by:

(Def. 3) ${}^{@}W = W$.

We now state the proposition

(9)[2] Every subset of W is a subset of V.

Let us consider K, V, W and let A be a subset of W. The functor ${}^{@}A$ yielding a subset of V is defined as follows:

(Def. 5)[3] ${}^{@}A = A$.

Let us consider K, V, W and let A be a non empty subset of W. Observe that ${}^{@}A$ is non empty.

One can prove the following propositions:

(10) $x \in \Omega_V$ iff $x \in V$.

(11) $x \in {}^{@}(\Omega_W)$ iff $x \in W$.

(12) $A \subseteq \Omega_{\mathrm{Lin}(A)}$.

(13) If $A \neq \emptyset$ and A is linearly closed, then $\sum l \in A$.

(15)[4] If $0_V \in A$ and A is linearly closed, then $A = \Omega_{\mathrm{Lin}(A)}$.

4. CYCLIC SUBMODULES

Let us consider K, V, a. Then $\{a\}$ is a subset of V.

Let us consider K, V, a. The functor $\prod^{*} a$ yields a strict subspace of V and is defined as follows:

(Def. 6) $\prod^{*} a = \mathrm{Lin}(\{a\})$.

5. INCLUSION OF LEFT R-MODULES

Let us consider K, M, N. The predicate $M \subseteq N$ is defined by:

(Def. 7) M is a subspace of N.

Let us note that the predicate $M \subseteq N$ is reflexive.

One can prove the following propositions:

(16) If $M \subseteq N$, then if $x \in M$, then $x \in N$ and if x is a vector of M, then x is a vector of N.

(17) Suppose $M \subseteq N$. Then $0_M = 0_N$ and if $m_1 = n_1$ and $m_2 = n_2$, then $m_1 + m_2 = n_1 + n_2$ and if $m = n$, then $r \cdot m = r \cdot n$ and if $m = n$, then $-n = -m$ and if $m_1 = n_1$ and $m_2 = n_2$, then $m_1 - m_2 = n_1 - n_2$ and $0_N \in M$ and $0_M \in N$ and if $n_1 \in M$ and $n_2 \in M$, then $n_1 + n_2 \in M$ and if $n \in M$, then $r \cdot n \in M$ and if $n \in M$, then $-n \in M$ and if $n_1 \in M$ and $n_2 \in M$, then $n_1 - n_2 \in M$.

(18) Suppose $M_1 \subseteq N$ and $M_2 \subseteq N$. Then

(i) $0_{(M_1)} = 0_{(M_2)}$,

(ii) $0_{(M_1)} \in M_2$,

(iii) if the carrier of $M_1 \subseteq$ the carrier of M_2, then $M_1 \subseteq M_2$,

(iv) if for every n such that $n \in M_1$ holds $n \in M_2$, then $M_1 \subseteq M_2$,

(v) if the carrier of $M_1 =$ the carrier of M_2 and M_1 is strict and M_2 is strict, then $M_1 = M_2$, and

(vi) $\mathbf{0}_{(M_1)} \subseteq M_2$.

[2] The proposition (8) has been removed.
[3] The definition (Def. 4) has been removed.
[4] The proposition (14) has been removed.

(21)[5] For all strict left modules V, M over K such that $V \subseteq M$ and $M \subseteq V$ holds $V = M$.

(22) If $V \subseteq M$ and $M \subseteq N$, then $V \subseteq N$.

(23) If $M \subseteq N$, then $\mathbf{0}_M \subseteq N$.

(24) If $M \subseteq N$, then $\mathbf{0}_N \subseteq M$.

(25) If $M \subseteq N$, then $M \subseteq \Omega_N$.

(26) $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$.

(27) $W_1 \cap W_2 \subseteq W_1$ and $W_1 \cap W_2 \subseteq W_2$.

(28) If $W_1 \subseteq W_2$, then $W_1 \cap W_3 \subseteq W_2 \cap W_3$.

(29) If $W_1 \subseteq W_3$, then $W_1 \cap W_2 \subseteq W_3$.

(30) If $W_1 \subseteq W_2$ and $W_1 \subseteq W_3$, then $W_1 \subseteq W_2 \cap W_3$.

(31) $W_1 \cap W_2 \subseteq W_1 + W_2$.

(32) $W_1 \cap W_2 + W_2 \cap W_3 \subseteq W_2 \cap (W_1 + W_3)$.

(33) If $W_1 \subseteq W_2$, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(34) $W_2 + W_1 \cap W_3 \subseteq (W_1 + W_2) \cap (W_2 + W_3)$.

(35) If $W_1 \subseteq W_2$, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(36) If $W_1 \subseteq W_2$, then $W_1 \subseteq W_2 + W_3$.

(37) If $W_1 \subseteq W_3$ and $W_2 \subseteq W_3$, then $W_1 + W_2 \subseteq W_3$.

(38) For all subsets A, B of V such that $A \subseteq B$ holds $\mathrm{Lin}(A) \subseteq \mathrm{Lin}(B)$.

(39) For all subsets A, B of V holds $\mathrm{Lin}(A \cap B) \subseteq \mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(40) If $M_1 \subseteq M_2$, then $\Omega_{(M_1)} \subseteq \Omega_{(M_2)}$.

(41) $W_1 \subseteq W_2$ iff for every a such that $a \in W_1$ holds $a \in W_2$.

(42) $W_1 \subseteq W_2$ iff $\Omega_{(W_1)} \subseteq \Omega_{(W_2)}$.

(43) $W_1 \subseteq W_2$ iff ${}^{@}(\Omega_{(W_1)}) \subseteq {}^{@}(\Omega_{(W_2)})$.

(44) $\mathbf{0}_W \subseteq V$ and $\mathbf{0}_V \subseteq W$ and $\mathbf{0}_{(W_1)} \subseteq W_2$.

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